On the computation of strain rate parameters – or – The rigorous character of some classical approximate formulas for strain rates

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Abstract: Strain rate parameters are derived for any epoch under the model of point motion with constant velocities from the direct differentiation and limit procedures on the rigorous formulas for strain parameters (dilatation, maximum shear strain, principal strains) introduced by Biagi & Dermanis. It is shown that the classical approximate formulas for strain rate parameters at an initial epoch based upon the infinitesimal strain tensor instead of the real one are in fact rigorous and identical with those derived rigorously! Strain rate parameters at any epoch are expressed as functions of those at the initial epoch and time.

1. Introduction

Deformation analysis on the horizontal plane is concerned with the point-wise comparison of the shape around each point at two epochs an initial (reference) one t_0 and a later (current) one t. It is based on either the deformation gradient $\mathbf{F}=\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}$ or the displacement gradient $\mathbf{J}=\frac{\partial \mathbf{u}}{\partial \mathbf{x}_0}=\mathbf{F}-\mathbf{I}$, where $\mathbf{x}_0$ and $\mathbf{x}$ are the point coordinates at epochs t_0 and t, respectively, while $\mathbf{u}=\mathbf{x}-\mathbf{x}_0$ is the point displacement. The computation of strain and strain rate parameters is traditionally carried out based on the infinitesimal strain matrix tensor $\mathbf{E}_{\text{inf}}=\frac{1}{2}(\mathbf{J}+\mathbf{J}^T)$, which is a first order approximation to the strain tensor $\mathbf{E}=\frac{1}{2}(\mathbf{F}^T\mathbf{F}-\mathbf{I})=\frac{1}{2}(\mathbf{J}+\mathbf{J}^T+\mathbf{J}^T\mathbf{J})$. Strain parameters are two-epoch quantities $p(t_0,t)$ while strain rate parameters can be either two-epoch quantities

$$\dot{p}(t_0,t)=\frac{d}{dt}p(t_0,t) \qquad (1)$$

or single epoch quantities

$$\dot{p}(t) = \lim_{\tau \to 0} \frac{d}{d\tau} p(t, t+\tau) = \lim_{t' \to t} \frac{d}{dt'} p(t, t') . \tag{2}$$

Here we will be mostly concerned with initial epoch strain rates

$$\dot{p}_0 = \dot{p}(t_0) = \lim_{t \to t_0} \frac{d}{dt} p(t_0, t) \tag{3}$$

for the special case where all points in the region move with constant velocities $\mathbf{v}$ according to the linear in time model

$$\mathbf{x}(t) = \mathbf{x}_0 + (t - t_0)\mathbf{v} . \tag{4}$$

which is the model currently implemented for the International Terrestrial Reference Frame (ITRF) see e.g. **Altammimi et al.** (2011).

In this case $\dot{\mathbf{J}} \equiv \frac{d\mathbf{J}}{dt} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}_0} \equiv \mathbf{L}$ is the velocity gradient.

Physically meaningful strain parameters are the dilatation Δ, the (maximum) shear strain γ and the principal strains $e_{\max}$ and $e_{\min}$. In the infinitesimal approach they are computed from the elements of the displacement gradient $\mathbf{J}$ according to the approximate formulas (**Malvern**, 1977, **Jaeger et al.**, 2007, **Dermanis** 2009)

$$\Delta = J_{11} + J_{22} , \tag{5}$$

$$\gamma_1 = J_{11} - J_{22} , \qquad \gamma_2 = J_{12} + J_{21} , \qquad \gamma = \sqrt{\gamma_1^2 + \gamma_2^2} , \tag{6}$$

$$e_{\max} = \frac{\Delta + \gamma}{2} , \qquad e_{\min} = \frac{\Delta - \gamma}{2} . \tag{7}$$

where γ_1, γ_2 are the so-called shear components. Due to linear character of some of the above relations, the corresponding initial epoch strain rate parameters are computed by replacing the elements of $\mathbf{J}$ with the corresponding elements of $\dot{\mathbf{J}} = \mathbf{L}$:

$$\dot{\Delta}_0 = L_{11} + L_{22} , \qquad \dot{\gamma}_{10} = L_{11} - L_{22} , \qquad \dot{\gamma}_{20} = L_{12} + L_{21} . \tag{8}$$

For the shear the two-epoch rate $\dot{\gamma}(t_0, t) = \frac{\gamma_1 \dot{\gamma}_1 + \gamma_2 \dot{\gamma}_2}{\gamma}$ has undefined limit for $t \to t_0$ since in this case both $\gamma_1 \to 0$, $\gamma_2 \to 0$ and $\gamma \to 0$. Instead the following relation is used

$$\dot{\gamma}_0 = \sqrt{\dot{\gamma}_{10}^2 + \dot{\gamma}_{20}^2} , \tag{9}$$

which at first sight appears to be unfounded. Furthermore

$$\dot{e}_{\max,0} = \frac{\dot{\Delta}_0 + \dot{\gamma}_0}{2} , \qquad \dot{e}_{\min,0} = \frac{\dot{\Delta}_0 - \dot{\gamma}_0}{2} . \tag{10}$$

We will show here that the above equations for strain rate parameters despite the fact that they are based on the approximate value $\mathbf{E}_{\text{inf}}$ of the strain tensor $\mathbf{E}$ they are rigorous formulas with no approximation involved! This will be achieved by applying the rigorous definition of initial epoch strain rate parameters on the rigorous formulas for strain parameters introduced by **Biagi & Dermanis** (2006, 2012). These are based on the deformation gradient matrix $\mathbf{F} = \dfrac{\partial \mathbf{x}}{\partial \mathbf{x}_0}$ which in the case of the constant velocity model for point motion becomes

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0} = \frac{\partial}{\partial \mathbf{x}_0}[\mathbf{x}_0 + (t - t_0)\mathbf{v}] = \mathbf{I} + (t - t_0)\frac{\partial \mathbf{v}}{\partial \mathbf{x}_0} \equiv \mathbf{I} + \tau \mathbf{L} \tag{11}$$

where $\mathbf{L} = \partial \mathbf{v} / \partial \mathbf{x}_0$ is the velocity gradient and we have also set $\tau = t - t_0$ for the shake of simplicity. The rigorous algorithm is as follows

$$\mathbf{C} = \mathbf{F}^T \mathbf{F} = \mathbf{I} + \tau(\mathbf{L} + \mathbf{L}^T) + \tau^2 \mathbf{L}\mathbf{L}^T =$$

$$= \begin{bmatrix} 1 + 2\tau L_{11} + \tau^2 (L_{11}^2 + L_{21}^2) & \tau(L_{12} + L_{21}) + \tau^2 (L_{11} L_{12} + L_{21} L_{22}) \\ \tau(L_{12} + L_{21}) + \tau^2 (L_{11} L_{12} + L_{21} L_{22}) & 1 + 2\tau L_{22} + \tau^2 (L_{12}^2 + L_{22}^2) \end{bmatrix}, \tag{12}$$

$$A = C_{11} + C_{22}, \qquad B = \sqrt{(C_{11} - C_{22})^2 + 4C_{12}^2}, \tag{13}$$

$$\lambda_1^2 = \frac{A + B}{2}, \qquad \lambda_2^2 = \frac{A - B}{2} \tag{14}$$

$$\gamma = \frac{\lambda_1 - \lambda_2}{\sqrt{\lambda_1 \lambda_2}}, \qquad \Delta = \lambda_1 \lambda_2 - 1 \tag{15}$$

$$e_{\max} = \frac{\lambda_1^2 - 1}{2}, \qquad e_{\min} = \frac{\lambda_2^2 - 1}{2}. \tag{16}$$

In this rigorous approach the principal strains $e_{\max}$ and $e_{\min}$ have been replaced by the equivalent principal linear elongation factors $\lambda_{\max} = \lambda_1$ and $\lambda_{\min} = \lambda_2$, occurring at two perpendicular principal directions. They are the maximum and minimum values of the linear elongation factor $\lambda = ds / ds_0$ for curves at any direction with initial length element ds_0 and final one ds.

2. Two-epoch strain rate parameters

The evaluation of two-epoch strain rate parameters $\dot{p}(t_0, t) = \dfrac{d}{dt} p(t_0, t)$ (**Dermanis**, 2010) follows from direct differentiation

$$\dot{\lambda}_1 = \frac{\dot{A}+\dot{B}}{4\lambda_1}, \qquad \dot{\lambda}_2 = \frac{\dot{A}-\dot{B}}{4\lambda_2}, \tag{17}$$

$$\dot{\gamma} = \frac{(\lambda_1+\lambda_2)(\lambda_2\dot{\lambda}_1 - \lambda_1\dot{\lambda}_2)}{2\lambda_1\lambda_2\sqrt{\lambda_1\lambda_2}}, \qquad \dot{\Delta} = \dot{\lambda}_1\lambda_2 + \lambda_1\dot{\lambda}_2, \tag{18}$$

$$\dot{e}_{\max} = \lambda_1\dot{\lambda}_1, \qquad \dot{e}_{\min} = \lambda_2\dot{\lambda}_2, \tag{19}$$

where dots denote derivatives with respect to time, $\dot{q} = \frac{dq}{dt}$.

Since these rates they will be the basis for the derivation of single epoch strain rates they must be expressed as functions of the elements of the velocity gradient $\mathbf{L}$. From (12) follows that

$$\dot{\mathbf{C}} = \mathbf{L} + \mathbf{L}^T + 2\tau\mathbf{L}\mathbf{L}^T =$$

$$\begin{bmatrix} \dot{C}_{11} & \dot{C}_{12} \\ \dot{C}_{12} & \dot{C}_{22} \end{bmatrix} = \begin{bmatrix} 2L_{11} + 2\tau(L_{11}^2 + L_{21}^2) & L_{12} + L_{21} + 2\tau(L_{11}L_{12} + L_{21}L_{22}) \\ L_{12} + L_{21} + 2\tau(L_{11}L_{12} + L_{21}L_{22}) & 2L_{22} + 2\tau(L_{12}^2 + L_{22}^2) \end{bmatrix} \tag{20}$$

while

$$\dot{A} = \dot{C}_{11} + \dot{C}_{22} \tag{21}$$

and

$$\dot{B} = \frac{(C_{11}-C_{22})(\dot{C}_{11}-\dot{C}_{22}) + 4C_{12}\dot{C}_{12}}{\sqrt{(C_{11}-C_{22})^2 + 4C_{12}^2}} = \frac{\dot{C}_{11}-\dot{C}_{22}}{\sqrt{1+\left(\frac{2C_{12}}{C_{11}-C_{22}}\right)^2}} + \frac{2\dot{C}_{12}}{\sqrt{1+\left(\frac{C_{11}-C_{22}}{2C_{12}}\right)^2}}. \tag{22}$$

3. Derivation of initial epoch strain rate parameters

In order to derive the limits of $\dot{\gamma}$ and $\dot{\Delta}$ as $t \to t_0$ ($\tau = t - t_0 \to 0$) we must first examine the same limits for $\dot{\mathbf{C}}$, $\dot{A}$, $\dot{B}$, as well as for λ_1 and λ_2. Obviously for $t \to t_0$ the corresponding limits are $\mathbf{F} \to \mathbf{I}$, $\mathbf{C} \to \mathbf{I}$ and $\lambda_1 \to 1$, $\lambda_2 \to 1$, since λ_1^2, λ_2^2 are the eigenvalues of $\mathbf{C}$. From (12) follows that

$$\frac{C_{11}-C_{22}}{2C_{12}} = \frac{2(L_{11}-L_{22}) + \tau(L_{11}^2 + L_{21}^2 - L_{12}^2 - L_{22}^2)}{2[(L_{12}+L_{21}) + \tau(L_{11}L_{12} + L_{21}L_{22})]}, \tag{23}$$

and

$$\lim_{t\to t_0} \frac{C_{11}-C_{22}}{2C_{12}} = \frac{L_{11}-L_{22}}{L_{12}+L_{21}}, \qquad \lim_{t\to t_0} \frac{2C_{12}}{C_{11}-C_{22}} = \frac{L_{12}+L_{21}}{L_{11}-L_{22}}. \tag{24}$$

From (20) follows that

$$\lim_{t\to t_0}\dot{\mathbf{C}} = \lim_{t\to t_0}\begin{bmatrix} \dot{C}_{11} & \dot{C}_{12} \\ \dot{C}_{12} & \dot{C}_{22} \end{bmatrix} = \begin{bmatrix} 2L_{11} & L_{12}+L_{21} \\ L_{12}+L_{21} & 2L_{22} \end{bmatrix} \tag{25}$$

As an immediate consequence

$$\lim_{t\to t_0}\dot{A} = \lim_{t\to t_0}(\dot{C}_{11}+\dot{C}_{22}) = 2(L_{11}+L_{22}) . \tag{26}$$

$$\lim_{t\to t_0}\dot{B} = \lim_{t\to t_0}\left[\frac{\dot{C}_{11}-\dot{C}_{22}}{\sqrt{1+\left(\frac{2C_{12}}{C_{11}-C_{22}}\right)^2}} + \frac{2\dot{C}_{12}}{\sqrt{1+\left(\frac{C_{11}-C_{22}}{2C_{12}}\right)^2}}\right] =$$

$$= \frac{2(L_{11}-L_{22})}{\sqrt{1+\left(\frac{L_{12}+L_{21}}{L_{11}-L_{22}}\right)^2}} + \frac{2(L_{12}+L_{21})}{\sqrt{1+\left(\frac{L_{11}-L_{22}}{L_{12}+L_{21}}\right)^2}}$$

$$= 2\sqrt{(L_{11}-L_{22})^2+(L_{12}+L_{21})^2} . \tag{27}$$

From the above limits follows that

$$\lim_{t\to t_0}\dot{\lambda}_1 = \lim_{t\to t_0}\frac{\dot{A}+\dot{B}}{4\lambda_1} = \frac{\lim_{t\to t_0}\dot{A}+\lim_{t\to t_0}\dot{B}}{4} = \frac{2(L_{11}+L_{22})+2\sqrt{(L_{11}-L_{22})^2+(L_{12}+L_{21})^2}}{4}, \tag{28}$$

$$\lim_{t\to t_0}\dot{\lambda}_2 = \lim_{t\to t_0}\frac{\dot{A}-\dot{B}}{4\lambda_2} = \frac{\lim_{t\to t_0}\dot{A}-\lim_{t\to t_0}\dot{B}}{4} = \frac{2(L_{11}+L_{22})-2\sqrt{(L_{11}-L_{22})^2+(L_{12}+L_{21})^2}}{4}. \tag{29}$$

Therefore

$$\lim_{t\to t_0}\dot{\gamma} = \lim_{t\to t_0}\frac{(\lambda_1+\lambda_2)(\lambda_2\dot{\lambda}_1-\lambda_1\dot{\lambda}_2)}{2\lambda_1\lambda_2\sqrt{\lambda_1\lambda_2}} = (\lim_{t\to t_0}\dot{\lambda}_1 - \lim_{t\to t_0}\dot{\lambda}_2) = \sqrt{(L_{11}-L_{22})^2+(L_{12}+L_{21})^2} \tag{30}$$

while

$$\lim_{t\to t_0}\dot{\Delta} = \lim_{t\to t_0}(\dot{\lambda}_1\lambda_2+\lambda_1\dot{\lambda}_2) = \lim_{t\to t_0}\dot{\lambda}_1 + \lim_{t\to t_0}\dot{\lambda}_2 = L_{11}+L_{22} . \tag{31}$$

Therefore the initial epoch rates $\dot{\gamma}_0 = \lim_{t\to t_0}\dot{\gamma}$ and $\dot{\Delta}_0 = \lim_{t\to t_0}\dot{\Delta}$ are given by

$$\dot{\gamma}_0 = \sqrt{(L_{11}-L_{22})^2+(L_{12}+L_{21})^2} , \qquad \dot{\Delta}_0 = L_{11}+L_{22} \tag{32}$$

which are identical with the formulas of the approximate infinitesimal approach! Indeed recalling that $\dot{\gamma}_{10} = L_{11} - L_{22}$ and $\dot{\gamma}_{20} = L_{12} + L_{21}$ we obtain

$$\dot{\gamma}_0 = \sqrt{\dot{\gamma}_{10}^2 + \dot{\gamma}_{20}^2} \,, \tag{33}$$

which is no other than the originally enigmatic relation (9)!
For the rates of the principal strains we have

$$\lim_{t \to t_0} \dot{e}_{\max} = \lim_{t \to t_0} (\lambda_1 \dot{\lambda}_1) = \lim_{t \to t_0} \dot{\lambda}_1 = \frac{(L_{11} + L_{22}) + \sqrt{(L_{11} - L_{22})^2 + (L_{12} + L_{21})^2}}{2} = \frac{\dot{\Delta}_0 + \dot{\gamma}_0}{2} \,, \tag{34}$$

$$\lim_{t \to t_0} \dot{e}_{\min} = \lim_{t \to t_0} (\lambda_2 \dot{\lambda}_2) = \lim_{t \to t_0} \dot{\lambda}_2 = \frac{(L_{11} + L_{22}) - \sqrt{(L_{11} - L_{22})^2 + (L_{12} + L_{21})^2}}{2} = \frac{\dot{\Delta}_0 - \dot{\gamma}_0}{2} \,. \tag{35}$$

Therefore the initial epoch rates $\dot{e}_{\max,0} = \lim_{t \to t_0} \dot{e}_{\max}$ and $\dot{e}_{\min,0} = \lim_{t \to t_0} \dot{e}_{\min}$ are given by

$$\dot{e}_{\max,0} = \frac{\dot{\Delta}_0 + \dot{\gamma}_0}{2} \,, \qquad \dot{e}_{\min,0} = \frac{\dot{\Delta}_0 - \dot{\gamma}_0}{2} \,, \tag{36}$$

which are identical with the supposedly approximate formulas of the infinitesimal approach!
We must remark that although point coordinates are linear functions of time the same is not true for the strain parameters and the strain rates cannot be used to induce strain parameters at any other epoch since $p(t_0, t) \neq p(t_0, t_0) + (t - t_0)\dot{p}_0$.

4. Strain rate parameters at any epoch

Although the derived strain rate parameters refer to the initial epoch t_0, the choice of t_0 itself is arbitrary and therefore the results are easily modified to apply to any epoch. For the derivation of single epoch strain rates $\dot{p}(t) = \lim_{t' \to t} \frac{d}{dt'} p(t, t')$ we note that the coordinate motion model $\mathbf{x}(t) = \mathbf{x}_0 + (t - t_0)\mathbf{v}$ gives

$$\mathbf{x}(t') = \mathbf{x}_0 + (t' - t_0)\mathbf{v} = \mathbf{x}(t) + (t' - t)\mathbf{v} \tag{37}$$

The deformation gradient for the two epochs t' and t is given by

$$\mathbf{F}(t, t') = \frac{\partial \mathbf{x}(t')}{\partial \mathbf{x}(t)} = \frac{\partial}{\partial \mathbf{x}(t)} \left[\mathbf{x}(t) + (t' - t)\mathbf{v} \right] = \mathbf{I} + (t' - t) \frac{\partial \mathbf{v}}{\partial \mathbf{x}(t)} \equiv \mathbf{I} + (t' - t)\mathbf{L}_t \,. \tag{38}$$

We note that while $\mathbf{F}(t_0, t) = \mathbf{I} + (t - t_0)\mathbf{L}$ now $\mathbf{F}(t, t') = \mathbf{I} + (t' - t)\mathbf{L}_t$, i.e. the initial epoch velocity gradient $\mathbf{L} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}_0}$ has been replaced by the instantaneous velocity gradient $\mathbf{L}_t = \dfrac{\partial \mathbf{v}}{\partial \mathbf{x}(t)}$. The structure of all strain rate parameters remains the same with respect to the relevant deformation gradient $\mathbf{F}$. Therefore since $\mathbf{F}(t_0, t)$ has been replaced by $\mathbf{F}(t, t')$ we simply need to replace $\mathbf{L}$ with $\mathbf{L}_t$ in the relevant formulas. In order to evaluate $\mathbf{L}_t$ in terms of $\mathbf{L}$ we note that

$$\mathbf{L}_t = \frac{\partial \mathbf{v}}{\partial \mathbf{x}(t)} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}_0}\frac{\partial \mathbf{x}_0}{\partial \mathbf{x}(t)} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}_0}\left[\frac{\partial \mathbf{x}(t)}{\partial \mathbf{x}_0}\right]^{-1} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}_0}\left[\mathbf{I} + \tau \frac{\partial \mathbf{v}}{\partial \mathbf{x}_0}\right] = \mathbf{L}(\mathbf{I} + \tau \mathbf{L})^{-1} \tag{39}$$

with $\tau = t - t_0$. Analytical inversion gives

$$(\mathbf{I} + \tau \mathbf{L})^{-1} = \begin{bmatrix} 1 + \tau L_{11} & \tau L_{12} \\ \tau L_{21} & 1 + \tau L_{22} \end{bmatrix}^{-1} = \frac{1}{D}\begin{bmatrix} 1 + \tau L_{22} & -\tau L_{12} \\ -\tau L_{21} & 1 + \tau L_{11} \end{bmatrix} \tag{40}$$

where

$$D = \det(\mathbf{I} + \tau \mathbf{L}) = (1 + \tau L_{11})(1 + \tau L_{22}) - \tau^2 L_{12} L_{21} = 1 + \tau (L_{11} + L_{22}) + \tau^2 (L_{11} L_{22} - L_{12} L_{21}) =$$

$$= 1 + \tau \,\mathrm{trace}\mathbf{L} + \tau^2 \det \mathbf{L} \tag{41}$$

The required velocity gradient becomes

$$\mathbf{L}_t = \begin{bmatrix} L_{t,11} & L_{t,12} \\ L_{t,21} & L_{t,22} \end{bmatrix} = \mathbf{L}(\mathbf{I} + \tau \mathbf{L})^{-1} = \frac{1}{D}\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}\begin{bmatrix} 1 + \tau L_{22} & -\tau L_{12} \\ -\tau L_{21} & 1 + \tau L_{11} \end{bmatrix} =$$

$$= \frac{1}{D}\begin{bmatrix} L_{11} + \tau (L_{11} L_{22} - L_{12} L_{21}) & L_{12} \\ L_{21} & L_{22} + \tau (L_{11} L_{22} - L_{12} L_{21}) \end{bmatrix} \tag{42}$$

The instantaneous strain rate parameters for an arbitrary epoch t follow by replacing in the formulas for the initial epoch rates the elements L_{ij} of $\mathbf{L}$ with the corresponding elements $L_{t,ij}$ of $\mathbf{L}_t$. Thus

$$\dot{\gamma}(t) = \sqrt{(L_{t,11} - L_{t,22})^2 + (L_{t,12} + L_{t,21})^2} = \frac{\dot{\gamma}_0}{D(t)}, \tag{43}$$

$$\dot{\Delta}(t) = L_{t,11} + L_{t,22} = \frac{\dot{\Delta}_0 + 2(t - t_0)\det \mathbf{L}}{D(t)}, \tag{44}$$

while

$$\dot{e}_{\max}(t) = \frac{\dot{\Delta}(t) + \dot{\gamma}(t)}{2} = \frac{e_{\max,0} + (t - t_0)\det \mathbf{L}}{D(t)}, \tag{45}$$

$$\dot{e}_{\min}(t) = \frac{\dot{\Delta}(t) - \dot{\gamma}(t)}{2} = \frac{e_{\min,0} + (t - t_0)\det \mathbf{L}}{D(t)} . \tag{46}$$

where since in view of (32) $\operatorname{trace}\mathbf{L} = L_{11} + L_{22} = \dot{\Delta}_0$

$$D(t) = 1 + (t - t_0)\dot{\Delta}_0 + (t - t_0)^2 \det \mathbf{L} . \tag{47}$$

5. Summary and conclusions

We have derived here three different types of rigorous strain rate parameters, which are the rates of corresponding strain parameters (dilatation, shear strain, principal strains). Since point-wise strain parameters describing deformation are functions of two epochs an initial one (reference shape) and a later one (current shape) it is possible to distinguish between:

(a) Two-epoch strain rates (rates with respect to the current shape)

(b) Single-epoch (instantaneous) strain rates at the initial (reference) epoch.

(c) Single-epoch (instantaneous) strain rates at any (current) epoch.

The most amazing result is the fact that the rigorous initial epoch strain rates (b above) are identical with the classical supposedly approximate strain rates which are derived on the basis of the infinitesimal approximation to the strain tensor. Moreover the problem of the enigmatic character of the formula for shear strain rate which does not comply with the rules of differentiation has been resolved.
Another interesting result is that strain rates at any current epoch (c above) are rational time functions involving the initial epoch rates.

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