On the analysis of the inverse gravimetric problem: Old and new results, open problems

F. Sansò

Politecnico di Milano, Polo Regionale di Como,
Via Valleggio 11 -Como, Italy

Introduction

The inverse gravimetric problem in its general form canbedefinedasfollows: given a simple surface S, which is the boundary, of a simply connected domain $B(S = \partial B)$ in R^3 and given in the exterior of S, denoted by Ω, a Newtonian potential, namely a function $u(x)$ harmonic in Ω and such that

$$u(\underline{x}) = O\left(\frac{1}{r}\right) \quad (r = |\underline{x}|), \tag{1.1}$$

to find a mass distribution $m(\underline{y})$, $\underline{y} \in \{B \cup S\}$, such that the Newtonian potential generated by $m(\cdot)$, i.e.

$$N(m) = \int_B \frac{1}{\ell_{xy}} dm(\underline{y}) \tag{1.2}$$

$$\ell_{xy} = |\underline{x} - \underline{y}|$$

coincides with u in Ω, namely

$$u(\underline{x}) = N(m) = \int_B \frac{1}{\ell_{xy}} dm(y), \quad \forall x \in \Omega , \tag{1.3}$$

This general formulation though, is not yet consistent from the mathematical point of view, if we don't add some restrictive conditions on the object $\{m(\underline{y})\}$ and on the meaning of formula (1.3). As a matter of fact, the family of functions of $\underline{y}$, indexed by $\underline{x} \in \Omega$,

$$\mathcal{HN} \equiv \left\{ \frac{1}{\ell_{xy}} 1;\ \underline{x} \in \Omega,\ \underline{y} \in \overline{B} \equiv B \cup S \right\}, \tag{1.4}$$

has specific properties which can be summarized as follows:

a) $\mathcal{HN}$ is a set of functions harmonic on B and, to be more precise, even on $\overline{B}$; as such they belong to $C^\infty(B)$,

b) if we assume that S is C^∞ too, e.g. there is a C^∞ function $c(\underline{x})$ positive in B and negative in Ω, so that

$$\{\underline{x} \in S\} \equiv \{\underline{x}\,;\; c(\underline{x}) = 0\}, \tag{1.5}$$

then the functions in $\mathcal{HN}$ are in $C^\infty(\bar{B})$ as well as in $C^\infty(S)$.

To be clear, let us remember that $f \in C^\infty$ on a closed set $\bar{B}$ means that there is an "extended" functions $\tilde{f}$ defined and C^∞ on an open set $O \supset \bar{B}$, such that $\tilde{f} \equiv f$ on $\bar{B}$.

In order to avoid further mathematical intricacies we shall always assume that $S \in C^\infty$ holds; in fact in examples through the paper we shall assume S to be a sphere of radius R, to make our reasoning more transparent.

We will denote by $\mathcal{LN}$ the linear span of $\mathcal{HN}$, namely the space of finite linear combination of elements of $\mathcal{HN}$,

$$\mathcal{LN} = \left\{ \sum_{k=1}^{n} a_k \ell_{x_k y}^{-1} \right\} \tag{1.6}$$

$$(a_k \text{ real}, \{x_k\} \in \Omega \quad k = 1, 2, \ldots, n)$$

Typically the linear manifold $\mathcal{LN}$ is viewed as a subspace of a larger space $H(\bar{B})$ endowed with some kind of topology and complete with respect to that (cf. [10]).

It is interesting to remark that, basically under very general hypotheses related to the Runge–Krarup theorem (cf. [5]), the closure of $\mathcal{LN}$ in $H(\bar{B})$ coincides with theintersection of $H(\bar{B})$ with $\mathcal{H}(B)$, the space ofallfunctions that are harmonic in B, open:

$$[\mathcal{LN}]_{H(\bar{B})} \equiv H(\bar{B}) \cap \mathcal{H}(B). \tag{1.7}$$

In any event, we shall assume that we want to perform our analysis for a space $H(\bar{B})$ for which (1.7) holds true and we shall call

$$HH(\bar{B}) = [\mathcal{LN}]_{H(\bar{B})}. \tag{1.8}$$

Put in this way, it is obvious that for (1.3) to be meaningful one has to assume that the object $\{m(\underline{y})\}$ is in the dual space $H'(\bar{B})$ of $H(\bar{B})$, i.e. it is a bounded/continuous linear functional on $H(\bar{B})$, so that its restriction to $HH(\bar{B})$ is meaningful too and one can write

$$\left\langle m(\underline{y}),\, \ell_{xy}^{-1} \right\rangle = \int_{\bar{B}} dm(y) = u(\underline{x}), \tag{1.9}$$

the integral notation having only a symbolic value, although it becomes totally adequate in the two best known examples, namely:

a) in case $H(\overline{B})=C(\overline{B})$ and $m(\underline{y})$ has then to be a Radon measure; then the integral (1.9) has to be understood in the sense of Stjelties–Lebesgue ([6]). In this case $HH(\overline{B})$ coincides with the Banach space of functions harmonic in *B*, and continuous up to the boundary,

b) in case $m(\underline{y})$ is absolutely continuous with respect to the Lebesgue measure and the density $\rho(\underline{y})$ defined by

$$dm(\underline{y})=\rho(\underline{y})d_3 y \tag{1.10}$$

is in $L^2(B)$, i.e. $H'(\overline{B})\equiv L^2(\overline{B})\equiv L^2(B)$: note that in this case, since $H'(\overline{B})$ is a Hilbert space it can be canonically identified with its dual, namely $H(\overline{B})=L^2(B)$. Moreover, in this example

$$HH(\overline{B})\equiv HL^2(B) \tag{1.11}$$

i.e. the space of functions harmonic in *B* and square integrable there, which is also called sometimes in literature a Hardy space (cf. [7]).

Other notable examples can be constructed with $\rho\in H^{s,2}(B)$, a Sobolev space with positive index s, although we shall not dwell on this case.

To perform an analysis of the inverse gravimetric problem means:

a) to define the space $H(\overline{B})$ and therefore $H'(\overline{B})$,

b) to verify that $H(\overline{B})\cap\mathcal{H}(B)$ is in fact a closed subspace of $H(\overline{B})$, that we denote $HH(\overline{B})$,

c) to find a decomposition of the space $H'(\overline{B})$ into two complementary spaces

$$H'(\overline{B})=K\oplus HH'(\overline{B}) \tag{1.12}$$

such that every "mass distribution" $m(\underline{y})$ can be uniquely expresses as the sum of one element $\rho_h(\underline{y})$ in a subspace $HH'(\overline{B})$, which is isomorphic to thedual of $HH(\overline{B})$, and one element $\rho_0(\underline{y})$ in the kernel of Newton's operator, i.e. such that

$$\left\langle \rho_0(\underline{y}),\ \ell_{xy}^{-1}\right\rangle\equiv 0 \quad \forall \underline{x}\in\Omega\ . \tag{1.13}$$

Furthermore the decomposition (1.13) has to be regular, namely the projector π of $H'(\overline{B})$ onto $HH'(\overline{B})$ has to be continuous.

The case $H(\overline{B})=L^2(B)$ is best understood and treated in geodetic literature; it will be recalled in §2 in a slightly more general form. In §3 the decomposition (1.13) will be made explicit for the $HL^2(\overline{B})$ case. In §4 examples are presented showing that there is a need for a more general theory to include the analysis of some classical cases.

2 The $L^2(B)$ case

As we have already stated, in this case

$$HH(\overline{B}) \equiv HL^2(B) \equiv HH'(\overline{B}) . \tag{2.1}$$

Note that by its proper definition, there is nodifference between $L^2(B)$ and $L^2(\overline{B})$ as the elements of L^2 are viewed as equivalence classes of functions coinciding almost everywhere in B (and therefore in $\overline{B}$).

Therefore in this case our Newton's equation

$$u(\underline{x}) \equiv \int_B \rho(\underline{y}) \ell_{xy}^{-1} d_3 x \tag{2.2}$$

is just interpreted as an $L^2(B)$ coupling.

To proceed with our analysis, we need to prove that the relation (1.7) holds true.

This can be done in several ways. We shall follow a two-step reasoning: first we show that $\mathcal{H}(B) \cap L^2(B)$ is a closed subspace of $L^2(B)$ namely $HL^2(B)$; then we shall prove that the family $\mathcal{HN}$ is total in $HL^2(B)$, i.e. that $\mathcal{LN}$ is dense in this space.

The first statement is one of the many forms of Harnack's theorem ([3]). In any event one elementary reasoning is as follows: let $\{u_n(\underline{x})\}$ be a convergent sequence in $\mathcal{H}(B) \cap L(B)$; then $\forall \underline{x} \in B$ we have for any ball $B(\underline{x}, \delta)$ with sufficiently small radius δ

$$u_n(\underline{x}) \equiv \frac{1}{B(\underline{x}, \delta)} \int_{B(\underline{x}, \delta)} u_n(\underline{y}) d_3 y \tag{2.3}$$

because of the theorem of the mean ([1]).

Call $u(\underline{x})$ the $L^2(B)$ limit of $\{u_n\}$; passing the limit in (2.3) we see that for almost every $\underline{x}$ and sufficiently small δ

$$u_n(\underline{x}) \equiv \frac{1}{B(\underline{x}, \delta)} \int_{B(\underline{x}, \delta)} u(\underline{y}) d_3 y , \tag{2.4}$$

i.e. it satisfies the mean property too. This means that $u(\underline{x})$ has to be harmonic as well (cf. [1]).

As for the second step we have basically to prove that if $\rho_h \in HL^2(B)$ and

$$\forall \underline{x} \in \Omega \quad u(\underline{x}) = \langle \ell_{xy}^{-1}, \rho_h(y) \rangle \equiv 0 , \tag{2.5}$$

then it is necessarily

$$\underline{y} \in B, \quad \rho_h(\underline{y}) = 0 . \tag{2.6}$$

To this aim, let us introduce a sphere S_e with radius R_e enclosing B. By combining $\{\ell_{xy}^{-1},\ \underline{x} \in S_e\}$ we can generate, e.g. through single layer potentials, all the functions $v(\underline{y})$ which are harmonic in a sphere concentric to S_e but with radius $R_e + \varepsilon$. Since this set of functions is dense in $HL^2(B)$ (in fact even in $\mathcal{H}(B)$) because of Runge – Krarup theorem ([5]), we can conclude that the same holds for $\mathcal{LN}$.

Up to here the points a) and b) of our analysis program have been settled, so we need only to define the kernel K of Newton's operator and verify that the corresponding projector $(I - \pi)$ is continuous.

Since we are conducting our analysis in a Hilbert space, namely $L^2(B)$, the choice of K is only natural, i.e. it consists in taking K as the orthogonal complement in $L^2(B)$ of $HL^2(B)$,

$$K = \left[HL^2(B) \right]^{\perp} . \tag{2.7}$$

In this case in fact π is an orthogonal projector a non-expanding operator with norm equal to 1.

So we finally obtain the decomposition

$$\forall \rho \in L^2(B), \quad \rho(\underline{y}) = \rho_h(\underline{y}) + \rho_0(\underline{y}) \tag{2.8}$$

with

$$\rho_h(\underline{y}) = \pi[\rho(\underline{y})] \tag{2.9}$$

$$\rho_0(\underline{y}) = (I - \pi)[\rho(\underline{y})] \tag{2.10}$$

and

$$\forall \underline{x} \in \Omega, \quad u(\underline{x}) \equiv \left\langle \ell_{xy}^{-1}, \rho \right\rangle = \left\langle \ell_{xy}^{-1}, \rho_h \right\rangle \tag{2.11}$$

$$\forall \underline{x} \in \Omega, \quad \left\langle \ell_{xy}^{-1}, \rho_0 \right\rangle = 0 . \tag{2.12}$$

How to characterize the operator π and the zero potential densities $\rho_0(\underline{y})$ will be object of next paragraph.

3 On the characterization of π and K

There are several ways to characterize the harmonic density $\rho_h(\underline{y})$, but one is particularly expressive in the frame of theoretical geodesy; namely π can be represented as an integral operator with a kernel $H(x, y)$ which is also the reproducing kernel of $HL^2(B)$. In fact, although $L^2(B)$ is not endowed with a reproducing kernel, its closed subspace $HL^2(B)$ is. This because the evaluation functional $\delta_{\underline{x}}(\underline{y})$ is indeed bounded in $HL^2(B)$, $\forall \underline{x} \in B$ (open) as a consequence of the majorization

$$\left|v(\underline{x})\right| \le C\|v\|_{L^2(B)}; \tag{3.1}$$

in (3.1) the constant C depends on the distance of $\underline{x}$ from the boundary (cf. [1]). Therefore there is a reproducing kernel $H(\underline{x}, \underline{y})$ such that

$$\forall v \in HL_2\left(\overline{B}\right), \quad v\left(\underline{x}\right) \equiv \left\langle H\left(\underline{x}, \underline{y}\right), \ v\left(\underline{y}\right)\right\rangle_{L^2}. \ \forall \underline{x} \in B \tag{3.2}$$

Since the behavior of v in B open completely characterizes this $L^2(B)$ function, (3.2) says that $\langle H(x,\cdot), \ \cdot\rangle$ is a representation of the identity in $HL^2(B)$.

Therefore if $\{e_k(\underline{x})\}$ is a complete orthonormal system in $HL^2(B)$, we can put

$$H\left(\underline{x}, \underline{y}\right) = \sum_{k=1}^{+\infty} e_k\left(\underline{x}\right) e_k\left(\underline{y}\right), \tag{3.3}$$

the series being $L^2(B)$ convergent. Indeed $H\left(\underline{x}, \underline{y}\right) = HL^2\left(B\right), \ \forall \underline{x} \in B$.

Now from the decomposition

$$\rho = \rho_h + \rho_0$$

we can take the scalar product

$$\left\langle H\left(\underline{x}, \underline{y}\right), \ \rho\left(\underline{y}\right)\right\rangle = \left\langle H\left(\underline{x}, \underline{y}\right), \ \rho_h\left(\underline{y}\right)\right\rangle + \left\langle H\left(\underline{x}, \underline{y}\right), \ \rho_0\left(\underline{y}\right)\right\rangle. \tag{3.4}$$

We note that

$$\left\langle H\left(\underline{x}, \underline{y}\right), \ \rho_h\left(\underline{y}\right)\right\rangle \equiv \rho h\left(\underline{x}\right)$$

because ρ_h is in $HL^2(B)$ and

$$\left\langle H\left(\underline{x},\underline{y}\right), \ \rho_0\left(\underline{y}\right)\right\rangle \equiv 0$$

as ρ_0 is in the orthogonal complement of $HL^2(B)$.

Therefore (3.4) becomes

$$\rho_h\left(\underline{x}\right) \equiv \left\langle H\left(\underline{x}, \underline{y}\right), \ \rho\left(\underline{y}\right)\right\rangle \tag{3.5}$$

proving that $\langle H(\underline{x},\cdot), \ \cdot\rangle$, viewed as on integral operator in $L^2(B)$, coincides with π.

Example As an exercise the reader can compute $H(\underline{x}, \underline{y})$ for the case that B is a sphere of radius R. In this case one can use as a complete orthogonal system in

$L^2(B)$ the functions $\left\{\left(\frac{r}{R}\right)^2 Y_{nm}(\sigma)\right\}$, which however are not normalized in $L^2(B)$. An easy computation shows that

$$\int d\sigma \int_0^R drr^2 \left(\frac{r}{R}\right)^{2n} Y_{nm}^2(\sigma) \equiv \frac{4\pi}{2n+3}.$$

Therefore we must have

$$H(\underline{x}, \underline{y}) = \frac{1}{4\pi} \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} \left(\frac{r_x r_y}{R^2}\right)^n (2n+3)\, Y_{nm}(\sigma_x)\, Y_{nm}(\sigma_y) =$$

$$= \frac{1}{4\pi} \sum_{n=0}^{+\infty} (2n+3)(2n+1) \left(\frac{r_x r_y}{R^2}\right)^n P_n(\cos\psi_{xy}), \qquad (3.6)$$

where ψ_{xy} is the spherical angle between x and y. In this case the sum of the series is easy to compute, starting from the identity

$$\sum_{n=0}^{+\infty} \xi^2(t) = [1 + \xi^4 + 2\xi^2 t]^{-1/2}$$

and results in

$$H(\underline{x}, \underline{y}) \equiv F(\xi, t) = \frac{3 - 10\xi^2 + 8\xi^3 t - \xi^4}{(1 + \xi^4 - 2\xi^2 t)^{5/2}}$$

where $\xi = \frac{r_x r_y}{R^2}$, $t = \cos\psi_{xy}$. Note how this kernel is regular in B but it becomes singular for $\underline{x} = \underline{y}$, $r_x - r_y \to R$.

We switch now to characterize the ρ_0 component of ρ, i.e. the elements of the subspace K.

As we have seen, the elements of K are such as tobe L^2-orthogonal to all L^2-harmonic functions and in particular to ℓ_{xy}^{-1}, so generating identically zero external potentials. Now there is a natural class of functions which accomplishes this task. Recall the definition of $\mathcal{D}(B)$, i.e. the space of functions which are C^∞ in B, so that they are identically zero close to the boundary S. Then, if $\varphi \in \mathcal{D}(B)$, $\Delta\varphi \in K$. In fact, by using a classical Green's identity, we have

$$\underline{x} \in \Omega:\ u(\underline{x}) = \int_B \ell_{xy}^{-1} \Delta\varphi(y) d_3 y = \int_S \left[t(\ell_{xy}^{-1}) n(\varphi) - n(\ell_{xy}^{-1}) t(\varphi) \right] dS \equiv 0 \qquad (3.7)$$

In (3.7) $t(v)$ denotes the trace operator on S, while $n(v)$ is the operator of the trace of the external normal derivative of v on S computed from the interior of B; (3.7) is true because by definition of φ

$$t(\varphi)\equiv 0, \quad n(\varphi)\equiv 0 . \tag{3.8}$$

We have established (3.7) for very smooth $\varphi \in (\mathcal{B})$, however for (3.7) to be meaningful we need only that $\Delta\varphi \in L^2(B)$, because we intend to require this to be the ρ_0 component of ρ; a natural choice is then to see $\mathcal{D}(B)$ as a subspace of $H^{2,2}(B)$, the space of functions which are square integrable in B together with their derivatives up to the second order. As it is known, this is a Hilbert space of Sobolev type. In particular it is perfectly known that the two operators $t(\ )$ and $n(\ )$ are bounded in the sense

$$\begin{aligned} t &: H^{2,2}(B) \to H^{3/2}(S) \\ n &: H^{2,2}(B) \to H^{1/2}(S), \end{aligned} \tag{3.9}$$

implying that the integral on S in (3.7) is certainly well-defined and continuous when φ varies in $H^{2,2}(B)$ (cf. [3]). Subsequently we can say that (3.7) holds for al φ in $H^{2,2}$ that can be reached by a sequence $\varphi_n \in \mathcal{D}(B)$ convergent in the $H^{2,2}$ topology. In other words (3.7) is true for all $\varphi \in [\mathcal{D}(B)]$, closed in $H^{2,2}$. Also this space is well-known in literature and usually called $H_0^{2,2}$ (cf. [3]). We could also say that (3.7) is true for all $\varphi \in H_0^{2,2}$, i.e. all $\varphi \in H^{2,2}$ satisfying (3.8).

The question then is whether

$$\Delta H\begin{pmatrix} 2,2 \\ 0 \end{pmatrix} \equiv K \tag{3.10}$$

or not; in other words, can all $\rho_0 \in K$ be expressed in the form

$$\rho_0 = \Delta\varphi, \quad \varphi \in H_0^{2,2} ?$$

The answer is in the affermative. Take a $\rho_0 \in K \subset L^2(B)$ and consider the B.V.P.

$$\begin{cases} \Delta\varphi = \rho_0 \\ t(\varphi) = 0 \end{cases} \tag{3.11}$$

As it is known (cf. [3]) there is one and only one solution, $\varphi \in H^{2,2}$, of (3.11). On the other hand, since $\rho_0 \in K$, we have from (3.7), recalling that $t(\varphi) = 0$,

$$u(\underline{x}) = \int_S t\left(\ell_{xy}^{-1}\right) n(\varphi)\, dS = \int_\Sigma \ell_{xy}^{-1} n(\varphi)\, dS \equiv 0 \quad \forall \underline{x} \in B . \tag{3.12}$$

Now $u(\underline{x})$ in (3.12) is a single layer potential, with density $n(\varphi) \in H^{1/2}(S) \subset L^2(S)$. Since such a potential is known to be everywhere continuous (cf. [3]), we can claim that $u(\underline{x}) = 0$ on S too. But then, if we penetrate B with $\underline{x}$, we find that $u(\underline{x})$ has to be harmonic in B and zero on S, i.e. $u(\underline{x}) = 0$ in B too. So $u(\underline{x}) = 0$ everywhere and this is enough to conclude that

$$n(\varphi)=0\,. \tag{3.13}$$

Therefore $\rho_0\in K$ implies that φ, solution of (3.11), is in $H_0^{2,2}$ as we wanted to prove.

Remark So far we have recovered and systematized a result known since long (cf. [8, 9]).

However wew ould also like to know, for the sake of future research, whether the representation

$$\rho=\rho_h+\Delta\varphi,\quad \varphi\in H_0^{2,2} \tag{3.14}$$

is unique or there is more freedom in choosing φ. After all the condition $t(\varphi)=0$ has been arbitrarily chosen. Let us call $HH^{2,2}(B)$ the subspace of $H^{2,2}(B)$ given by

$$HH^{2,2}(B)=H^{2,2}(B)\cap\mathcal{H}(B); \tag{3.15}$$

since $H^{2,2}$-convergence of a sequence f_n to f implies also L^2-convergence of first and second derivatives, it is easy to see that $HH^{2,2}(B)$ is a closed subspace of $H^{2,2}(B)$. Then the following Lemma holds.

Lemma (of decomposition) given any $\rho\in L^2(B)$, the following decomposition holds

$$\begin{cases}\rho=\rho_h+\rho_0\\ \rho_h\in HL^2(B),\quad \rho_0\in K\end{cases}, \tag{3.16}$$

$$\begin{cases}\rho=\Delta(\psi+v)\\ \psi\in H_0^{2,2}(B),\quad v_0\in H^{2,2}(B)\end{cases}, \tag{3.17}$$

where ρ_h, ρ_0, ψ are unique, v is arbitrary.

Proof: that (3.16) holds we already know; that (3.17) holds for a unique ψ, with $v\equiv 0$, we also know.

That $\Delta(\psi+v)=\Delta\psi=\rho_0$ if $v\in HH^{2,2}(B)$ is trivial, i.e. $\varphi=\psi+v$ is again an admissible function to represent $\rho_0=\Delta\varphi$. That (3.17) is the most general decomposition is also obvious because if one has to have

$$\Delta(\psi+v)=\rho_0=\Delta\psi$$

then it must be $\Delta v=0$ too. □

So, not only the subspace $H_0^{2,2}$ is suitable to represent $\rho_0\in K$, but also all the linear manifolds obtained by translating $H_0^{2,2}$ along a function harmonic in B.

4 Is the L^2 analysis sufficient?

Naturally the analysis just developed is a pure mathematical tool, useful to clarify what is the state of knowledge on the interior density of a body when we know its exterior potential.

Most interesting it is then to introduce further physical information, whether in deterministic or in stochastic form, to be combined with the potential and to restrict the class of meaningful solutions. Nevertheless, not even the pure mathematical analysis has still accomplished its job, because there are quite meaningful mass distributions used in classical analysis that do not enter int the L^2 densities case.

Such are for instance point masses, single layers, double layers etc. None of them possess an L^2 density, but the first two cases are examples of measures, while the third is a distribution, with some surface in $\overline{B}$ (e.g. S) as support.

All these cases are still manageable in relatively easy way if the support of $m(y)$, be it in general a distribution, is contained in B open, but they require finer analysis tools if the support reaches the boundary S.

Let us illustrate this by the example of a body B which is a ball of radius R, and a mass distribution which is a point mass M placed at distance $R_0 < R$ from the center on the z axis.

Example Let's take B as a sphere with radius R and as mass distribution a point mass at P_0, the point at distance R_0 along the z axis; the mass at P_0 is M. As it is known $\forall r = |\underline{x}| > R$ we have a potential $u_0(\underline{x})$ that can be expanded into the series

$$u_0(\underline{x}) = \frac{GM}{R_0} \sum_{u=0}^{+\infty} \left(\frac{R_0}{r}\right)^{n+1} \frac{Y_{no}(\sigma_x)}{\sqrt{2n+1}} \tag{4.1}$$

in fact remember that

$$Y_{no}(\sigma_x) = \sqrt{2n+1}\, P_n(\cos\vartheta_x),$$

with ϑ_x the spherical colatitude of $\underline{x}$.

As we see, the mass distribution generating the potential is

$$m_0(\underline{y}) = M\delta(\underline{y} - \underline{x}_0) \tag{4.2}$$

where $\delta(\cdot)$ is not a function and in particular it is not square integrable on B. However it is obvious that there is a unique L^2 density equivalent to $\{m_0(y)\}$ outside B. In fact our potential can be equivalently generated by a uniform density $\overline{\rho}_\varepsilon$ on a small sphere of radius ε, such that $B(\underline{x}_0, \varepsilon) \subset B$, (i.e. $R_0 + \varepsilon < R$), if the totalmass of this small sphere is the same as M. Since such a density is in $L^2(B)$ it must also have a projection ρ_h on $HL^2(B)$. We want to find the explicit form of ρ_h. Let us put

$$\rho_h(\underline{y}) = \sum_{n=0}^{+\infty} \sum_{m=-n}^{h} \rho_{nm} \left(\frac{S}{R}\right)^n Y_{nm}(\sigma_y) \tag{4.3}$$

An elementary computation shows that the potential $u(\underline{x})$, generated for $r = |\underline{x}| > R$ by the density(4.3), is

$$u(\underline{x}) = 3\frac{BG}{R} \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} \frac{\rho_{nm}}{(2n+1)(n+3)} \left(\frac{r}{R}\right)^{n+1} Y_{nm}(\sigma_x) \tag{4.4}$$

$$\left(B = \frac{4}{3}\pi R^3\right).$$

Equating (4.1) and (4.4) we find

$$\rho_{nm} = \frac{1}{3}\frac{M}{B}(2n+3)\sqrt{2n+1}\left(\frac{R_0}{R}\right)^n \delta_{mo}\,. \tag{4.5}$$

As we can see, the density ρ_h we found is in $L^2(B)$, i.e.

$$\int_B \rho_h^2(\underline{y}) s^2 ds d\sigma_y = \frac{4\pi R^3}{9}\frac{M^2}{B^2}\sum_{n=0}^{+\infty}(2n+3)(2n+1)\left(\frac{R_0}{R}\right)^{2n} \tag{4.6}$$

is bounded on condition that

$$R_0 < R\,. \tag{4.7}$$

However, (4.6) says also that

$$\lim_{R_0 \to R} \|\rho_h\|^2_{L^2(B)} = +\infty$$

showing the difficulty of using L^2 theory when the mass distribution concentrates on the boundary.

The example can be easily generalized to arbitrary smooth surfaces S and to very irregular mass distributions on conditions, as we already said, that they have a support in B open. On the contrary, treating distributions with support that reaches the boundary S, is matter for a future finer analysis.

References

[1] Axler S., Bourdon P. and Ramey W. (1992) Harmonic Function Theory, Springer-Verlag

[2] Barzaghi R. and Sansò F. (1988) Remarks on the inverse gravimetric problem. Geoph. J. of the RAS 92:3

[3] Gilbarg David, Trudinger Neil S. (1983) Elliptic Partial Differential Equations of Second Order. Grund. der Math. Wiss. 225 Springer-Verlag

[4] Kirsch A. (1996) An introduction to the mathematical theory of inverse problems. Springer-Verlag

[5] Krarup T. (1969) A contribution to the mathematical foundation of physical geodesy. Kobenhavn, Geodaetisk institut, 1969.

[6] Riesz F. and Sz.-Nagy B. (1965) Lecons d'analyse fonctionnelle Gauthier-Villars

[7] Rudin W. (1981) Real and complex analysis. Tata-McGraw Hill

[8] Sansò F. (1980) Internal collocation, Mere. Accad. Naz. Lincei, Ser. VIII,. XVI

[9] Vanicek P. and Kleursberg A. (1975) What an external gravitational potential can really tell us about mass distribution. Boll. Geof. Teor. Appl. Vol. XXCII, No. 108

[10] Yosida K. (1980) Functional Analysis. Springer-Verlag