Helmert's Transform by Quaternions. A Revisitation

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1. Introduction

The problem of connecting the Cartesian coordinates of a set of physical points $\{P_i,\ i=1,2,...,N\}$ seen in two different reference frames (R) and (S), with algebraic vector components $\mathbf{r}=(x,y,z)^T$ and $\mathbf{s}=(x',y',z')^T$, is known to be solved, when the scale of the two systems is not the same, by a Helmert transformation, also called S-transformation, S standing for similarity, namely

$$\mathbf{s}=\mathbf{t}+\lambda U\mathbf{r} \tag{1}$$

where $\mathbf{t}$ is the translation between (R) and (S), with components in (S)

U is a proper rotation matrix, namely an isometric matrix with determinant equal to 1

λ is a scale parameter, namely the ratio between the unit length of (S) and the unit length of (R).

The above definitions imply

$$U^T U=UU^T=I\,,\quad (\text{in } R^3) \tag{2}$$

$$\det U=1 \tag{3}$$

$$\lambda\geq 0 \tag{4}$$

In particular (3) implies that there are not an odd number of reflections, or that U represents a rotation topologically connected to the identity, i.e. belonging to the connected subgroup of the unitary group which contains I too.

How to realize the transformations is the direct Helmert's problem, solved by formula (1).

How to derive $(\mathbf{t},U,\lambda)$ from known triples of coordinates $\mathbf{s}_i$, $\mathbf{r}_i$ respectively in (S) and (R), is the inverse Helmert problem.

The problem is quite relevant in Geodesy at both global and local level; at global level to align large networks adjusted each in its own reference frame, at local level to adjust for instance a local GNSS network to a national reference frame (see [1], [3]).

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Not to be mentioned the use of (1) in photogrammetry, where in fact the first exact solution has matured, [7].

Indeed the degrees of freedom contained in an S-transform are 7, so a limited information on coordinates in (R) and (S) can in principle be used to determine the corresponding parameters. Yet, since the known coordinates $\{\mathbf{s}_{oi}\}$ and $\{\mathbf{r}_{oi}\}$ are usually affected by errors in the respective reference frames, a redundant number of points is used, and a l. s. approach is often applied as estimation algorithm. This is the practice applied for instance in the determination of ITRF ([1]). The problem is indeed nonlinear in several senses, first of all the representation of U in terms of 3 parameters, typically Euler angles, is highly non-linear. Second, the product of λ by U introduces another non-linearity in the dependence on the parameters. Third, the product of the "observed" vectors $\{\mathbf{r}_{oi}\}$ by λU is indeed a nonlinearity for a general adjustment framework, as noticed by A. Pope long ago ([4]).

A standard approach to non-linear l. s. problems is to linearize, starting from approximate values of the parameters, and then iterate. Apart form the fact that a proof of the convergence is missing, in many cases finding "approximate values" for the parameters is not a straightforward task. Consider for instance the case that you have to adapt an optical model to the real object when the two are in a general position in space and scales have completely different values.

So the possibility of finding exact solutions of the relevant l. s. problem, at least under some simplifying hypotheses, is intrinsically interesting.

A first solution, assuming that $\{\mathbf{s}_{oi}\}$ are affected by errors, while $\{\mathbf{r}_i\}$ are errorless has been found by J. M. Tienstra. The assumption on the stochastic model of the errors $\mathbf{v}_i = \mathbf{s}_{oi} - \mathbf{s}_i$ ($\mathbf{s}_i$ values) was simply

$$E\{\mathrm{v}_i \mathrm{v}_k^T\} = \sigma_i^2 I \delta_{ik} \ . \tag{5}$$

It is interesting to observe that the success of the method depends on the idea that the full matrix U should be considered as a parameter and equation (2) should be added as a side condition, without going into the matter of the complicated representation of U in terms of Euler (or Cardan) angles.

A second independent solution was then found by the author ([5]), under the same simplifying hypotheses on the stochastic model, exploiting Hamilton's quaternion formalism. An approach this that is more used in navigation than in geodesy and photogrammetry. To let the reader to get acquainted with the quaternion algebraic rules and 3D rotations, in §2 we shortly review the useful relations used in [5].

Stated in these terms, however, the method has a limited use, specially because to assume that one of the two systems of coordinates, normally $\{\mathbf{r}_i\}$, is devoid of errors, is too restrictive.

For instance, if one could enhance the model assuming that

$$\begin{cases} \mathrm{v}_i = \mathbf{s}_{oi} - \mathbf{s}_i \\ E\{\mathrm{v}_i\} = 0,\ E\{\mathrm{v}_i \mathrm{v}_k^T\} = \sigma_i^2 \boldsymbol{I}\, \delta_{ik} \end{cases} \tag{6}$$

and that

$$\begin{cases} \boldsymbol{\varepsilon}_i = \mathbf{r}_{oi} - \mathbf{r}_i \\ E\{\boldsymbol{\varepsilon}_i\} = 0,\ E\{\boldsymbol{\varepsilon}_i \boldsymbol{\varepsilon}_k^T\} = \omega_i^2 \boldsymbol{I}\, \delta_{ik} \end{cases} \tag{7}$$

as well as

$$E\{\mathbf{v}_i \boldsymbol{\varepsilon}_k^T\} = 0 \tag{8}$$

the author believes that the method could be applied, at least for a speedy solution, to a much larger number of cases. This will be done in §3 of the paper. A few remarks follow in §4.

2. Quaternions and 3Drotations

A quaternion q is an element of $R^4 \equiv (R \otimes R^3)$ that we shall write in the form

$$q = q_o + i\mathbf{q} \quad ; \tag{9}$$

q_o is called the real part of $q, \mathbf{q} \in R^3$ is the imaginary (or vector) part of q, while i is meant to behave like the imaginary unit, i.e.

$$i^2 = -1 \ . \tag{10}$$

Indeed R^4, with its representation (9), is a vector space with respect to ordinary sum and multiplication by a real number. What is interesting, is that we can make of R^4 a non-commutative algebra $\mathcal{H}$ (stemming for Hamilton who first investigated the matter) with the product definition

$$q = q_o + i\mathbf{q}\ ,\ \ p = p_o + i\mathbf{p}\ ;\ \ pq = p_o q_o - \mathbf{q}\cdot\mathbf{p} + i(q_o\mathbf{p} + p_o\mathbf{q} + \mathbf{q}\wedge\mathbf{p})\ . \tag{11}$$

As we see, $pq \neq qp$ because of the presence of the vector product $\mathbf{q}\wedge\mathbf{p}$ which is anti-commutative. Nevertheless, the product (1) is associative.

We define conjugate $\overline{q}$ of q, modulus of q, $|q|$ and inverse of q, q^{-1} as

$$\overline{q} = q_o + i\mathbf{q} \tag{12}$$

$$|q| = \sqrt{q_o^2 + |\mathbf{q}|} = \sqrt{q\overline{q}} = \sqrt{\overline{q}q} \tag{13}$$

$$q^{-1} = |q|^{-2}\, \overline{q} \ ; \tag{14}$$

the reader is invited to verify that $\overline{q}$ is an involution, i.e. $\overline{\overline{q}} = q$, $|q|$ is a nonnegative number and $|q|>0$ if $q\neq 0$, q^{-1} is both left and right inverse, i.e. $q^{-1}q=qq^{-1}=1$.

Other properties of the product (11), useful in the present context and easily verified are

$$\Re q=\Re\overline{q} \tag{15}$$

$$\Re pq=\Re qp \tag{16}$$

$$\overline{qp}=\overline{p}\,\overline{q} \tag{17}$$

The following exercise of differentiation will also be useful.

$$\begin{aligned} d(q\overline{q}) &= dq\overline{q}+qd\overline{q}=(qd\overline{q})+qd\overline{q}= \\ &=2\Re(qd\overline{q}) \end{aligned} \tag{18}$$

Finally, we note that if the differential form $\Re(qd\overline{q})$ is zero $\forall dq$, then

$$p_o\, dq+\mathbf{p}\cdot d\mathbf{q}=0 \quad \forall dq_o,\ d\mathbf{q}\Rightarrow p=0\ . \tag{19}$$

Now we notice that the following “projection” relations hold

$$\begin{cases} \{\Re q;\ p\in\mathcal{H}\}\equiv R \\ \{Imq=\mathbf{q};\ q\in\mathcal{H}\}\equiv R^3, \end{cases} \tag{20}$$

which can be seen as vector subspaces of R^4. Moreover, we introduce the following quaternion transformation

$$s=qr\overline{q} \tag{21}$$

and observe that if $r\in R^3$ (i.e. r is pure imaginary $r=i\mathbf{r}$), then s is imaginary too. In fact

$$s+\overline{s}=qr\overline{q}+qr\overline{q}=qr\overline{q}+q(-r)\overline{q}=0\ ,$$

because for every imaginary r we have $r=-\overline{r}$.

Moreover, we can compute

$$\begin{aligned} |s|^2=s\overline{s} &= (qr\overline{q})(\overline{qr\overline{q}})=qr\,|q|^2\overline{rq}= \\ &=|q|^2q\,|r|^2\overline{q}=|q|^4|r|^2; \end{aligned} \tag{22}$$

it follows that if we choose q on the unit sphere in R^4, namely

$$|q|^2=q_o^2+|\mathbf{q}|^2=1 \tag{23}$$

then $|s|^2=|r|^2$. A transformation of this kind can only be a product of a rotation by a reflection of one or more axes; the reflection of two axes, say x and y, is in fact a rotation of π around z, so (21) can only be a rotation by a reflection of one or three axes. We want to prove that in fact (21) is a proper rotation and even more that every rotation can be presented in the form (21). The representation is not exactly one to one because q and $-q$ represents the same transformation (21). So we show that in fact when q spans the unit sphere in R^4, the transformation (21) covers twice the subgroup of proper rotations in the unitary group. To prove the above statement we will use a particular representation valid for all rotations in 3D.

We start observing that every rotation is in fact a rotation around an axis, identified by a unit vector $\mathbf{u}$ (see [6]). Let α be the angle of the rotation around $\mathbf{u}$; looking at Fig. 1 one recognizes that the component $(\mathbf{r}\cdot\mathbf{u})\mathbf{u}$ is invariant; on the contrary, the component $\mathbf{r}-(\mathbf{r}\cdot\mathbf{u})\mathbf{u}=\mathbf{r}_0$ is rotated by α in the plane orthogonal to $\mathbf{u}$, till it reaches $\mathbf{r}_0'$.

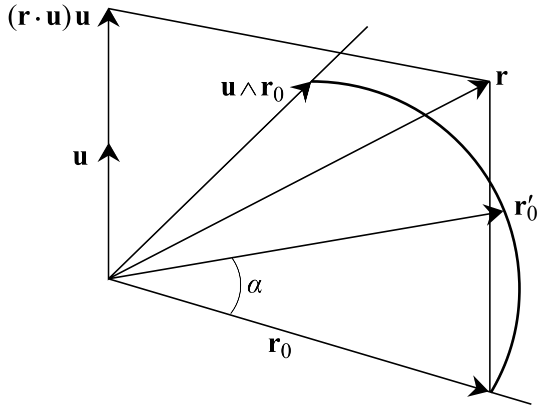

Figure 1: *The geometry of the rotation of* $\mathbf{r}$ *around* $\mathbf{u}$ *by an angle α.*

We note that, since $\mathbf{u}$ has modulus 1 and is orthogonal to $\mathbf{r}_0$, one has

$$|\mathbf{u}\wedge\mathbf{r}_o| = |\mathbf{r}_0'| = |\mathbf{r}_0| \ . \tag{24}$$

So one has

$$\mathbf{r}_0' = \mathbf{r}_0\cos\alpha + \mathbf{u}\wedge\mathbf{r}_0\sin\alpha \ . \tag{25}$$

We observe as well that

$$\mathbf{u}\wedge\mathbf{r}_o = \mathbf{u}\wedge[\mathbf{r}-(v\cdot\mathbf{u})\mathbf{u}] = \mathbf{u}\wedge\mathbf{r} \ . \tag{26}$$

Therefore one obtains

$$\begin{aligned} U\mathbf{r} &= (\mathbf{r}\cdot\mathbf{u})\mathbf{u}+\mathbf{r}_0' = (\mathbf{r}\cdot\mathbf{u})\mathbf{u}+\mathbf{r}_o\cos\alpha+\mathbf{u}\wedge\mathbf{r}\sin\alpha = \\ &= \mathbf{r}\cos\alpha+(1-\cos\alpha)(\mathbf{r}\cdot\mathbf{u})\mathbf{u}+\sin\alpha\,\mathbf{u}\wedge\mathbf{r}. \end{aligned} \tag{27}$$

The relation (27) is a general representation of a 3D rotation. We can observe that also (27) is not unique, since the transformation $(\mathbf{u},\alpha)\rightarrow(-\mathbf{u},-\alpha)$ produces the same rotation.

In any way, we can return to (21) and perform the triple product, applying the rule (11). After some algebra, one obtains

$$q r \bar{q} = i\,[(q_o^2 - |\bar{q}|^2)\,\mathbf{r} + 2(\mathbf{r}\cdot\mathbf{q})\mathbf{q} + 2q_o\mathbf{q}\wedge\mathbf{r}] \ . \tag{28}$$

Now, recalling (23), we see that we can put

$$\begin{cases} q_o = \cos\vartheta \\ \mathbf{q} = \sin\vartheta\mathbf{v} \end{cases} \tag{29}$$

with $\mathbf{v}$ a unit vector, directed as $\mathbf{q}$ and a suitable ϑ.

With the above position, (21) becomes

$$q r \bar{q} = i\,[(\cos^2\vartheta - \sin^2\vartheta)\,\mathbf{r} + 2\sin^2\vartheta(\mathbf{r}\cdot\mathbf{v})\mathbf{v} + 2\sin\vartheta\cos\vartheta\mathbf{v}\wedge\mathbf{r}] \,. \tag{30}$$

Since

$$\cos^2\vartheta - \sin^2\vartheta = \cos 2\vartheta \,,\ \ 2\sin^2 = 1 - \cos 2\vartheta \,,\ \ 2\sin\vartheta\cos\vartheta = \sin 2\vartheta \,.$$

(30) can be written as well

$$q r \bar{q} = i\,[\cos 2\vartheta \mathrm{r} + (1-\cos 2\vartheta)\,(\mathbf{r}\cdot\mathbf{v})\mathbf{v} + \sin 2\vartheta\mathbf{q}\wedge\mathbf{r}] \,. \tag{31}$$

A comparison between (27) and (31) shows that by choosing

$$\vartheta = \frac{\alpha}{2} \ ,\ \mathbf{v} = \mathbf{u} \tag{32}$$

we can reproduce any proper 3D rotation, i.e. the operation (21), $R^3 \to R^3$, is a representation of the group of proper rotations. Even more, since it is clear that in (27) $(\mathbf{u}, \alpha)$ can always be chosen in a way that $0 \le \alpha \le \pi$, the corresponding quaternion constructed with the rules (29) and (32), will have a real part

$$w_o = \cos\frac{\alpha}{2} \ge 0 \ \ ;$$

in this way we wipe away the ambiguity between the choice q , $-q$ to represent the same rotation.

3. The l. s. estimate of (t, λ, q)

The model of the Helmert transformation by using the identification $R^3 \equiv ImH$ and the representation (21) of the rotation is

$$s = t + \lambda q r \bar{q} \tag{33}$$

where (t,λ,q) are the unknown parameters. The observational model is now

$$\begin{cases} s_{oj} = s_j + v_j \\ E\{v_j\} = 0 \ , \ E\{\Re v_j \bar{v}_k\} = 3\delta_{jk}\sigma_j^2 \end{cases} \tag{34}$$

$$\begin{cases} r_{oj} = r_j + \varepsilon_j \\ E\{\varepsilon_j\} = 0 \ , \ E\{\Re \varepsilon_j \bar{\varepsilon}_k\} = 3\delta_{jk}\omega_j^2 \end{cases} \tag{35}$$

with

$$s_j = t + \lambda q r_j \bar{q} \tag{36}$$

The factor 3 comes from the fact that $v_j\bar{v}_j = |v_j|^2 = v_{xj}^2 + v_{yj}^2 + v_{zj}^2$ and each of the components has variance σ_j^2 according to (6); the same holds for $\varepsilon_j\bar{\varepsilon}_j = |\varepsilon_j|^2$.

Moreover, v_j and ε_k are all independent of one another.

Under the conditions (6), (7), (8) the l. s. principle to estimate (t,λ,q) corresponds to searching for the minimum of the function

$$\bar{F}(r_j,...,s_j,...,t,\lambda,q) = \sum_{j=1}^{N} \frac{|v_j|^2}{\sigma_j^2} + \frac{|\varepsilon_j|^2}{\omega_j^2} \tag{37}$$

with the side conditions (36) and (23). A reduced, but equivalent, approach (see for instance [2]) consists of eliminating r_j, s_j by (34), (35) from the unknown parameters, insert them into (33) that becomes

$$s_{oj} - t - \lambda q r_{oj}\bar{q} = v_j - \lambda q \varepsilon_j \bar{q} = \eta_j \ . \tag{38}$$

We note that, considering the stochastic model of v_j, ε_j, we have that η_j are independent for different indexed *j* and

$$\begin{aligned} \{\eta_j\bar{\eta}_j\} &= E\{v_j\bar{v}_k\} + \lambda^2 E\{q\varepsilon_j\bar{q}q\bar{\varepsilon}_j\bar{q}\} = \\ &= E\{v_j\bar{v}_k\} + \lambda^2 E\{\varepsilon_j\bar{\varepsilon}_j\} = \sigma_j^2 + \lambda^2\omega_j^2 = \gamma_j \ . \end{aligned} \tag{39}$$

We call

$$w_j = \gamma_j^{-1} \tag{40}$$

and remark that indeed both γ_j, w_j depend on λ. The reduced l. s. principle then becomes the search for the minimum of

$$F(t,\lambda,q) = \sum_{j=i}^{N} w_j |\eta_j|^2 = \sum_{j=i}^{N} w_j |s_{oj} - t - \lambda q r_{oj}\bar{q}|^2 + \alpha q\bar{q} \tag{41}$$

where we have introduced the Lagrange multiplier α to take into account, later, the side condition (23).

Two remarks are in order.

Remark 1. *Since one might be willing, at the end of the procedure, to get the estimates of* r_j, s_j*, i.e. of* ε_j, v_j*, we notice that from the relation*

$$\eta_j = v_j - \lambda q \varepsilon_j \overline{q} ,$$

the best estimates of v_j, ε_j *given* η_j *are provided by the prediction formulas*

$$\tilde{v}_j = \frac{\sigma_j^2}{\gamma_j} \eta_j \tag{42}$$

$$\tilde{\varepsilon}_j = -\lambda \frac{\omega_j^2}{\gamma_j} \eta_j . \tag{43}$$

These are in fact the BLUE of v_j, ε_j *when* η_j *are "observed" and* λ *is known (see [2]). So a straightforward estimate of* v_j, ε_j *can be obtained from the l. s. estimates* $\hat{\lambda}, \hat{\eta}$*.*

Indeed (42), (43), considered as functions of η_j and λ are non-linear and then substituting the l.s. estimators might not be optimal. Yet it is reasonable specially considering that $\tilde{v}_j, \tilde{\varepsilon}_j$ are linear functions of η_j and for $\hat{\lambda}$ the following Remark 2 holds.

Remark 2. *We notice that* $F(t, \lambda, q)$ *depends on* λ *through both, the residuals* η_j*, and the weights* $w_j = (\sigma_j^2 + \lambda^2 \omega_j^2)^{-1}$*.*

However, we know that the l. s. estimator has a weak dependence on the weights. Therefore the use of a reasonable estimate of λ *in* $w_j(\lambda)$ *is not significantly affecting the result. On the other hand, also our knowledge of* σ_j^2 *and* ω_j^2 *is typically not very precise. Since a good estimator of* λ *can be derived, in any event, from the relation*

$$| s_j - b_s | = \lambda | r_j - b_r | \tag{44}$$

$$b_s = \frac{1}{N} \sum_{j=1}^{N} s_j , \quad b_r = \frac{1}{N} \sum_{j=1}^{N} r_j \tag{45}$$

Suggesting

$$b_s = \frac{1}{N} \sum_{j=1}^{N} \frac{| s_{oj} - b_{so} |}{| r_{oj} - b_{ro} |} , \tag{46}$$

we shall assume that λ in $w_j(\lambda)$ is known, i.e. w_j themselves are known numbers. This produces a tremendous simplification of the analytic treatment, so we will accept the above hypothesis. Let us observe that, since we are going in any way to obtain a further estimator $\hat{\lambda}$ from the l. s. procedure, an obvious iteration process can be established. A fast convergence should be achieved in view of the above reasoning.

Given the two Remarks, we can pass to the minimization of (41), where w_j are taken as fixed numbers.

We start minimizing (41) with respect to t; one has

$$dF = d(\sum w_j \eta_j \overline{\eta}_j) = 2\Re \sum w_j \eta_j (-d\overline{t}) = 0 , \tag{47}$$

implying

$$\sum w_j \eta_j = \sum w_j s_{oj} - \sum w_j t - \lambda q (\sum w_j r_{oj}) \overline{q} \tag{48}$$

Introducing the weighted barycentres

$$\begin{cases} b_{so} = \dfrac{\sum w_j s_{oj}}{\sum w_j} \\ b_{ro} = \dfrac{\sum w_j r_{oj}}{\sum w_j} \end{cases} , \tag{49}$$

(48) gives

$$t = b_{so}^{w} - \lambda q b_{ro}^{w} \overline{q} . \tag{50}$$

It is convenient now to introduce (50) into the observation equations, so that calling

$$\delta s_{oj} = s_{oj} - b_{so} , \ \ \delta r_{oj} = r_{oj} - b_{ro} \tag{51}$$

one can write

$$\eta_j = \delta s_{oj} - \lambda q \delta r_{oj} \overline{q} . \tag{52}$$

In this way (41) is reduced to a function of λ, q only, namely

$$F = \sum_{j=1}^{N} w_j | \delta s_{oj} - \lambda q \delta r_{oj} \overline{q} |^2 + \alpha q \overline{q} . \tag{53}$$

It is somehow clever to develop the modulus square in (53), to arrive at

$$F = \sum w_o | \delta s_{oj} |^2 + \lambda^2 \sum w_j | \delta r_{oj} |^2 + 2\lambda \Re \sum_j w_j \delta s_{oj} q \delta r_{oj} \overline{q} + \alpha q \overline{q} . \tag{54}$$

To obtain (54), the relation (15), (16) have been used, together with the fact that for every imaginary quaternion v one has $\overline{v} = -v$.

The minimization of F with respect to λ is now elementary and given by

$$\lambda = \frac{\Re \sum_{j=1}^{N} w_j \delta s_{oj} q \delta r_{oj} \overline{q}}{\sum_{j=1}^{N} w_j |\delta r_{oj}|^2} \tag{55}$$

Now we go for the minimization of (56) with respect to q. We have, after some algebra,

$$\begin{aligned} F &= 2\lambda \Re \sum w_j \delta s_{oj} dq \delta r_{oj} \overline{q} + 2\lambda \Re \sum w_j \delta s_{oj} q \delta r_{oj} \overline{q} + \alpha(dq\overline{q} + qd\overline{q}) = \\ &= \Re[(4\lambda \sum w_j \delta s_{oj} q \delta r_{oj} + 2\alpha q)\delta \overline{q}] = 0 \end{aligned}$$

namely

$$\sum w_j \delta s_{oj} q \delta r_{oj} = -\frac{\alpha}{2\lambda} q \ . \tag{56}$$

The equation (56) hides in fact an eigenvalue equation for a 4×4 matrix A. In order to get the explicit expression of A one can, recalling that $\delta s_j, \delta r_j$ are pure imaginary, exploit the product

$$\begin{aligned} \delta s\, q\, \delta r = &-\delta \mathbf{s} \cdot \delta \mathbf{r} q_o + (\delta \mathbf{s} \wedge \delta \mathbf{r})\mathbf{q} + \\ &+ i\{(\delta \mathbf{s} \wedge \delta \mathbf{r}) q_o - [\delta \mathbf{r}(\delta \mathbf{s} \cdot \mathbf{q}) + \delta \mathbf{s}(\delta \mathbf{r} \cdot \mathbf{q})] + (\delta \mathbf{s} \cdot \delta \mathbf{r})\mathbf{q}\} \end{aligned} \tag{57}$$

Introducing (57) into (56) and equating real and imaginary parts one gets the wanted system of 4 equations into the 4 components of q.

The exercise is left to the reader.

The eigenvalue problem has in general 4 eigenvalues Λ_i and 4 eigenvectors q_i; since the modulus of the eigenvectors is not fixed, we can always satisfy condition (23) by choosing the eigenvector with unitary modulus.

The choice of the right q among the 4 is easily accomplished if we note that, given q we can reckon the target function by (53). The eigenvector that minimizes F is the right one.

Even more, since from (56) with $\Lambda = -\alpha / 2\lambda$ known and q the sought quaternion, we derive

$$\Re \sum_{j=1}^{N} w_j \delta s_{oj} q \delta r_{oj} \overline{q} = \Lambda$$

we find that (55) can be substituted by the simpler equation

$$\lambda = \frac{A}{\sum_{j=1}^{N} w_j |\delta r_{oj}|^2} \tag{58}$$

Remark 3. *As we have already noticed, from (55) we get a new estimate of* λ*. One can then check whether it agrees with the prior* λ *given by (47). Would this not happen in a satisfactory way, one could recompute* w_j *and iterate.*

4. Discussion

The result of §3, is that the problem of finding the S-transformation between two reference frames, in which a set of points has observed coordinates, can be treated by the quaternion approach if the simple stochastic structure (34), (35) can be hypothesized. This permits to handle different accuracies of coordinates of the various points, but not a correlation between points, nor between the coordinates of the same point. The author has tried various generalizations without success. There seems to be a tradeoff between the complexity of the stochastic structure and the complexity of the corresponding quaternion representation. Despite the above limitations, the author believes that being capable of producing a solution without any prior information on the rotation, the present quaternion algorithm is interesting, if not for any other reason because it is capable of producing a solution once we accept to simplify the stochastic model so as to became compliant with (34), (35). As a further proof of the above statement, let us assume that the stochastic model of s_{oj}, r_{oj} is simplified in a way that

$$E\{|v_j|^2\} = \sigma_j^2 = \sigma^2, \quad E\{|\varepsilon_j|^2\} = \omega_j^2 = \omega^2 \tag{59}$$

i.e. σ_j^2, ω_j^2 do not depend on the point index j. Then we have

$$w_j = w(\lambda) = \sigma^2 + \lambda^2\omega^2 \tag{60}$$

independent of j, and the target function (41) becomes

$$F = w(\lambda)\sum_{j=1}^{N} |s_{oj} - t - \lambda q r_{oj}\overline{q}|^2 + \alpha q\overline{q}\,. \tag{61}$$

The minimization with respect to t proceeds as before, giving

$$t = b_{so} - \lambda q b_{ro}\overline{q} \tag{62}$$

where now (take $w_j = w$ in (49))

$$b_{so} = \frac{1}{N}\sum_{j=1}^{N} s_{oj}\ , \quad b_{ro} = \frac{1}{N}\sum_{j=1}^{N} r_{oj} \quad . \tag{63}$$

So we can define

$$\delta s_{oj} = s_{oj} - b_{so}\ , \quad \delta r_{oj} = r_{oj} - b_{ro} \tag{64}$$

and returning to (61) we get

$$\begin{aligned} F &= w(\lambda)\sum_{j=1}^{N} |\delta s_{oj} - \lambda q r_{oj}\overline{q}|^2 + \alpha q\overline{q} = \\ &= w(\lambda)[\sum_{j=1}^{N} |\delta s_{oj}|^2 + \lambda^2 \sum_{j=1}^{N} |\delta r_{oj}|^2 + 2\lambda \Re \sum_{j=1}^{N} \delta s_{oj} q \delta r_{oj}\overline{q}] + \alpha q\overline{q}\ . \end{aligned} \tag{65}$$

The variation of (65) with respect to q gives

$$2\lambda w(\lambda)\sum_{j=1}^{N} \delta s_{oj} q \delta r_{oj} + \alpha q\ , \tag{66}$$

which is basically the same eigenvalue equation as (56), but with constant weights. Namely we can write

$$\sum_{j=1}^{N} \delta s_{oj} q \delta r_{oj} = \Lambda q \quad , \tag{67}$$

giving 4 λ 's and 4 q 's. In particular, for each of the 4 q 's we have the relation

$$\Re \sum_{j=1}^{N} \delta s_{oj} q \delta r_{oj} \overline{q} = \Lambda$$

so that for each solution (Λ, q), (65) becomes

$$F = w(\lambda)[\sum_{j=1}^{N} |\delta s_{oj}|^2 + \lambda^2 \sum_{j=1}^{N} |\delta r_{oj}|^2 + 2\lambda\Lambda] + \alpha\ . \tag{68}$$

We can now minimize (68) with respect to λ. We leave the algebra to the reader and report the result obtained, after some simplifications,

$$\omega^2 \Lambda \lambda^2 + (\omega^2 \sum_{j=1}^{N} |\delta s_{oj}|^2 + \sigma^2 \sum_{j=1}^{N} |\delta r_{oj}|^2\ \lambda)\lambda - \sigma^2 \Lambda = 0 \quad . \tag{69}$$

As we see, equation (69) has always one positive and one negative root; indeed the positive root is the only interesting to us. So from (67) and (69) we have 4 solutions (q, Λ, λ) that used in (68) will tell us which one is the good. As we see therefore, a simplification of the stochastic model leads to a completely manageable problem.

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