Preface

This paper illustrates the use of GRAVSOFT for geoid determination in the mountains, with comparisons to high-quality GPS-leveling data in an international test data case. It represents a typical application of the GRAVSOFT set of programs, which prof. Dimitris Arabelos have worked with and influenced through many years, a.o. through frequent discussions and requests at his extended research visits to Denmark (especially at the former Geodetic Institute in the late 1980’s, where many of the GRAVSOFT modules where developed), as well as research discussions in Thessaloniki and elsewhere.

Abstract

In this paper a geoid computation using the remove-restore technique and spherical FFT is used to derive a geoid of the Auvergne region, central France. Long and medium wavelengths are removed by the EGM08 spherical harmonic model, while terrain reduction is handled by the RTM method, implemented by prism integration and Fourier techniques. The results are compared to accurate 1st order GPS-leveling data, showing an r.m.s. accuracy of 29 mm in the best case, which is at the best reported level for the Auvergne test data set. The geoid comparison results vary depending on how the topography is handled relative to the inferred EGM08 topography. Due to some inherent problems in implementing the RTM method for highly varying reference topography, it appears that best results are obtained for a relatively low-resolution (30°) reference heights, irrespective of whether EGM08 is used at a corresponding resolution (360) or to full resolution.

1. Introduction

A major goal for physical geodesy has been the determination of the geoid at an accuracy of 1 cm, matching the accuracy of GPS height determination. Although the cm-geoid has been demonstrated in numerous cases in lowland areas with dense gravity data coverage, combining gravity and (sparse) GPS-levelling data to make an operational “GPS geoid”, so far no convincing case of attaining a cm-geoid in mountainous regions has been reported. This is likely a consequence of
both insufficient gravity data coverage, theoretical shortcomings, and – especially – insufficient quality of the leveling data, used to compute “ground truth” geoid (or quasi-geoid) values by

\[ N = h - H \quad \text{or} \quad \zeta = h - H^* \]  

Here \( N \) is the classical geoid height, \( \zeta \) the height anomaly (“geoid at the surface of the terrain”), \( h \) the ellipsoidal height of GPS, and \( H \) and \( H^* \) the orthometric and normal heights, respectively.

A “best” comparison data set for geoid determination methods in the mountains was distributed in 2008 by the late H. Duquenne, IGN, France, on behalf of the International Geoid Service (IGeS), as a “ground truth” example for precise geoid determination method. A number of international groups have already worked with this data set and reported results using different methods (for a summary see, e.g, Bazarghi and Sanso, 2009).

The Auvergne data set consist of about 240,000 gravity data points from the Bureau Gravimétrique (provided by the French geological agency BRGM), covering a \( 6^\circ \times 8^\circ \) area including most of France; a DEM based on 3” SRTM height data covering a somewhat larger \( 8^\circ \times 10^\circ \) area; and a set of 75 GPS-levelling points in the “Massif Centrale” area, all with 1st order leveling connections, and a quoted GPS ellipsoidal height accuracy of 2-3 cm (RBF points) or “slightly better” (NIVAG points), cf. Fig. 1 (H. Duquenne, pers.comm.). The elevations of the GPS-leveling points range from 206 to 1235 m, and the highest mountain in the central area is 1886 m; it is therefore a relatively moderate mountainous area.

**Fig. 1:** Left: data coverage for geoid computation. Right: distribution of GPS-leveling data in the central area.
2. Geoid computation by the remove-restore method: Methodology

The methodology for geoid construction is based on remove-restore techniques. The anomalous gravity potential $T$ is split into three parts:

$$T = T_{EGM08} + T_{RTM} + T_{res}$$  \hspace{1cm} (2)

where $T_{EGM08}$ is the contribution of the EGM08 spherical harmonic expansion (Pavlis and Kenyon, 2008), of form

$$T_{EGM08} = \frac{GM}{r} \sum_{n=2}^{N} \sum_{m=0}^{n} \left( \frac{R}{r} \right)^{n} \left( C_{nm} \cos \lambda + S_{nm} \sin \lambda \right) P_{nm}(\sin \varphi)$$  \hspace{1cm} (3)

EGM08 is here used to maximal degree either 360 or 2190. $T_{RTM}$ are the terrain effects, and $T_{res}$ the residual gravity field. $T$ is treated as a spatial function, and with the French national height system being normal heights, I will in the sequel work consistently with the quasigeoid as the fundamental “geoid”, i.e.

$$\zeta = \frac{T(\varphi, \lambda, H^*)}{\gamma_0}$$  \hspace{1cm} (4)

where $\gamma_0$ is the normal gravity, and $\varphi$ and $\lambda$ the geographical latitude and longitude. Another geoid-like surface to be used is the quasi-geoid at sea level

$$\zeta^* = \frac{T(\varphi, \lambda, 0)}{\gamma_0(\varphi)}$$  \hspace{1cm} (5)

$\zeta^*$ is the “geoid” corresponding to the harmonically downward continued anomalous potential $T$, neglecting any mass associated with the topography. It is therefore not the same as the classical geoid $N$, which corresponds to the actual equipotential surface inside the mass. The separation of the geoid and the quasigeoid is to first order expressed by the Bouguer anomaly $\Delta g_B$

$$\zeta - N = -\frac{\Delta g_B}{\gamma_0} H$$  \hspace{1cm} (6)

whereas the separation between $\zeta^*$ and the quasigeoid $\zeta$ to first order involves the free-air anomaly (or, as a slightly better approximation, the gravity disturbance $\delta g$)

$$\zeta - \zeta^* \approx \frac{\delta g}{\gamma_0} H$$  \hspace{1cm} (7)

The above equations follows easily from the basic Chapter 8 on Molodensky’s theory in Heiskanen and Moritz (1967). Using equation (6) or (7) is therefore
straightforward in geoid determination computations to either work with $\zeta$ or $N$; it is a simple task to convert between the two quantities, as long as the first-order approximation can be accepted.

For the terrain-reduction, the terrain effects are reduced relative to a mean elevation surface, as shown schematically in Fig. 2. The terrain potential is subtracted from the observations using a prism integration, i.e. representing the mass between the actual topography and the mean elevation surface (m.e.s.) as mass prisms of either positive or negative density, nominally 2.67 g/cm$^3$. In practice the geoid “restore” signal is computed by Fourier methods, for details see Forsberg (1984, 1985).

In the RTM method the resolution of the m.e.s. is controlled by the user, through a suitable low-pass filter. Ideally such a filter should ideally correspond to the equivalent resolution of the highest spherical harmonic expansion degree $N$ in the reference potential. This has worked well with e.g. reference fields to degree and order 360 (such as EGM96).

The prism implementation of the RTM method has an inherent problem: the method leaves a point $P$ above the m.e.s. in the mass-free domain, whereas a point $Q$ below the m.e.s. after the reduction would correspond to the value inside the reference topography mass, cf. Fig. 2.

![Fig. 2: Principle of RTM reduction. The mean elevation surface is computed by low-pass filtering of a DEM.](image)

Since all geodetic gravity field modeling methods require observations derived from a harmonic function, i.e. in a mass-free environment above the geoid, a correction must be made after the prism computations. This correction – the harmonic correction – to first order only applies to gravity anomalies, and only for points below the m.e.s. It has the magnitude

$$\Delta g_{hc} = -4\pi G \rho (h_Q - h_{ref})$$

(8)

where $G$ is the gravitational constant, and $\rho$ the density; for details see Forsberg (1984). The equation (8), equivalent to the Prey reduction of sub-surface gravimetry, relies on a Bouguer plate assumption for the reference topography, and there-
fore breaks down for a too quickly varying reference topography (as in the case of the EGM08 resolution of 5'). What is needed is a series expansion of the harmonic correction, taking into account the non-level surface of the m.e.s., and being able to compute values anywhere in the space between the m.e.s. and the zero level. This problem has to my knowledge not been solved, in spite of the similarity to the general “Molodensky” downward continuation problem, and the theoretical advantage of using reference topography, not real topography.

To overcome the harmonic correction problem, alternative formulations for the RTM method in the frequency domain may be used; such methods have e.g. been used in the spectral combination method of the European geoid (Denker and Torge, 1998), and also in the derivation of EGM08 itself, which is based on an RTM-like interpolation of gravity field effects from a global 15’ grid to a 5’ grid over large parts of the continents (as a consequence of lack of data or classification of the available 5’ gravity grid data). An alternative RTM approach, based on multipole mass expansions, have been proposed by Vermeer and Forsberg (1992). None of these alternative methods have been used here, so the RTM-reductions used in the Auvergne tests are based on the “simple” prism method with the harmonic correction (8).

With the EGM08 and RTM terrain effects removed, the geoid contribution is found by Fourier implementation of Stokes/Molodensky’s formula

\[ \zeta_{\text{res}} = \frac{R}{4\pi} \int \int (\Delta g_{\text{res}} + g_1^\ell) S(\psi) d\sigma \]  

where \( S \) is Stokes’ function, modified as outlined below.

When the RTM reduction is used, the Molodensky \( g_1^\ell\)-term will generally be insignificant (Forsberg and Sideris, 1989), and the formula converts into the conventional Stokes’ formula for the geoid, in principle applied to gravity field quantities at the geoid (i.e, \( \zeta^* \) and \( \Delta g^* \)). The evaluation of the Stokes formula is done using spherical Fourier techniques, for a details of the methods see Schwarz et al (1990) and Forsberg and Sideris (1993). In the method the Stokes integral is evaluated by a series of convolutions, each accurate around a certain reference latitude \( \phi \)

\[ z_{\text{res}} = S_{\text{ref}}(D_j, D_l)^*[D g_{\text{res}}(j, l) \sin j] = F^{-1} \left[ F(S_{\text{ref}}) F(D g_{\text{res}} \sin j) \right] \]

where \( F[\cdot] \) is the two-dimensional Fourier transform.

To keep the inherently highly accurate GRACE gravity field information in EGM08 to be “overruled” by the influence from terrestrial gravity data, modified Stokes functions are used. The modified Wong-Gore formulation used here is of form

\[ S_{\text{mod}}(\psi) = S(\psi) - \sum_{n=2}^{N_\Delta} a(n) \frac{2n+1}{n-1} P_n \cos(\psi) \]
where the “GRACE-transition” coefficient $\alpha(n)$ increase linearly from 0 to 1 between degrees $N_1$ and $N_2$

$$a(n) = \begin{cases} 
1 & \text{for } 2 \leq n \leq N_1 \\
\frac{N_2 - n}{N_2 - N_1} & \text{for } N_1 \leq n \leq N_2, \ n = 2, \ldots, N \\
0 & \text{for } N_2 \leq n
\end{cases}$$

(12)

The estimation of $N_1$ and $N_2$ can only be done empirically, but values in the range 60-90 would be expected, based on the error characteristics of GRACE.

3. The Auvergne gravimetric geoid computation

The GRAVSOFT system (Tscherning et al, 2002) implements the different procedures outlined in section 2. Before the computations a slightly thinned gravity data of 59097 points were selected (pixel binning to approx. 1 km resolution), and at the same time the DEM was thinned to 6” resolution in latitude and longitude for ease of handling.

The EGM08 was computed to either degree 360 or 2190 in sandwich grid form (Fig. 3), so that the gravity and potential (i.e., $\Delta g$ and $\zeta$) could be evaluated anywhere in space by three-dimensional grid interpolation; the values computed in this way are thus harmonic continued values inside the terrain mass, not the physical values.

Fig. 3: Principle of EGM08 sandwich grid interpolation. From the dense grids the full three-dimensional field in the relevant height ranges are computed by linear interpolation.

The RTM terrain effects were computed either with a m.e.s. of resolution 5’ or 30’, with the m.e.s. simply constructed by a Gaussian filter with a corresponding half-width resolution. The mean elevation surfaces are shown in Fig. 4. Terrain effects were subsequently subtracted from the gravity data, with statistical results shown in Table 1. It is seen that the available gravity data show a very good fit to EGM08, with little bias. The larger standard deviation seen for the degree 360 use of EGM08 (and the corresponding lower resolution m.e.s.) is in principle showing the magnitude of the non-terrain related gravity field signal in the spherical harmonic band between 360 and 2190.
Table 1: Statistics of the gravity data reductions (mgal)

<table>
<thead>
<tr>
<th></th>
<th>EGM degree 2190 / RTM 5'</th>
<th>EGM degree 360 / RTM 30&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>No of gravity points</td>
<td></td>
<td></td>
</tr>
<tr>
<td>59097</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>4.2</td>
<td>4.2</td>
</tr>
<tr>
<td>Std. dev.</td>
<td>22.8</td>
<td>22.8</td>
</tr>
<tr>
<td>Free-air – EGM08</td>
<td>−2.5</td>
<td>−3.5</td>
</tr>
<tr>
<td>Mean</td>
<td>11.6</td>
<td>17.9</td>
</tr>
<tr>
<td>Std. dev.</td>
<td>0.3</td>
<td>8.2</td>
</tr>
<tr>
<td>Free-air – EGM08 – RTM</td>
<td>0.3</td>
<td>0.6</td>
</tr>
</tbody>
</table>

Fig. 4: RTM reference elevations: Left: 5', right 30' resolution. Colour scale 0 to 1500 m.

The reduced gravity data were subsequently gridded in the data coverage region (Fig.1) to a grid of 0.01° × 0.0125° resolution by least squares collocation. The spherical FFT conversion was done using zero-padding in a zero-padded grid of dimension 1200 x 1280 data points, using 3 reference latitude parallels. The RTM geoid terrain effects were subsequently restored using ζ-terrain effects computed by FFT in a similar grid, and finally the gravimetric geoid was obtained by adding the geoid EGM08 effects, computed at the level of the topographic surface. Fig 5 shows the magnitude of the geoid “restore” effects for the “full” EGM08 solution (with 5’ resolution RTM) from the terrain and the residual gravity. The r.m.s. values of the terrain effects were 19 cm and 3 cm r.m.s., for the 30’ and 5’ RTM reductions, respectively, illustrating the different frequency content.

The outcome of the above processing scheme is in principle the quasigeoid ζ, when the Stokes function (9) is evaluated directly with the reduced gravity anomalies Δg_{res}. In principle a more rigorous estimate of the quasigeoid would involve a downward continuation of Δg_{res} to the ground level by Δg_{res}^* ≈ Δg_{res} − T_{zz}^*h, with the second order gradient T_{zz} estimated Δg_{res} by Fourier methods, and the subsequent conversion of ζ^* to ζ by (7); this Molodensky-style principle is im-
implemented in one of the GRAVSOFT programs (geofour) but tests did not show any significant improvement.

![Image](image.png)

**Fig. 5:** The geoid “restore” signals for EGM08 to degree 2190. Left: RTM terrain effect (colour scale -20 to 40 cm). Right: Geoid effect from residual gravity data (colour scale -20 to 20 cm).

### 4. Comparison to GPS-levelling

The GPS-levelling data were used first to estimate an “optimal” Wong-Gore modification of Stokes’ function. This was done for the EGM08 field used to degree 360, and 30' RTM. The results are shown in Table 2. It is seen that the data supports an optimal degree band of degree 80-90, in good agreements with the estimated accuracy of GRACE.

**Table 2:** Statistics of fit of the geoid to GPS-leveling geoid heights for different kernel modification parameters.

<table>
<thead>
<tr>
<th>Modification degree band</th>
<th>Mean (m)</th>
<th>Std.dev. (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>None (normal Stokes)</td>
<td>−0.176</td>
<td>0.034</td>
</tr>
<tr>
<td>Degree 40-50</td>
<td>−0.084</td>
<td>0.035</td>
</tr>
<tr>
<td>Degree 60-70</td>
<td>−0.100</td>
<td>0.036</td>
</tr>
<tr>
<td>Degree 80-90</td>
<td>−0.138</td>
<td>0.029</td>
</tr>
<tr>
<td>Degree 100-110</td>
<td>−0.149</td>
<td>0.041</td>
</tr>
<tr>
<td>Degree 120-130</td>
<td>−0.147</td>
<td>0.057</td>
</tr>
</tbody>
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Rene Forsberg
Table 3 shows the geoid fits for the 75 GPS-levelling points, for both EGM08 alone (to degree 2190), the EGM08 (degree 360) gravimetric geoid solution, the EGM08 quasigeoid fit, as well as a geoid solution using the full EGM08 to degree 2190, but a 30'-resolution RTM surface (i.e., taking the shorter-wavelength topography into account twice). It is seen that the latter geoid model actually gives the best r.m.s. fit of 29 mm, indicating that the “double accounting” of the topography does not matter in practice (which it should not in principle, since the remove-restore principle will account for this). That the RTM 5’ model seems to fit poorer is likely due to the approximation errors in the harmonic correction.

Table 3: Comparison of different geoid models to GPS-leveling

<table>
<thead>
<tr>
<th>Geoid model fit, 75 points (unit: m)</th>
<th>Mean</th>
<th>Std.dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>EGM08 to degree 360</td>
<td>-.109</td>
<td>0.172</td>
</tr>
<tr>
<td>EGM08 to degree 2190</td>
<td>-.109</td>
<td>0.036</td>
</tr>
<tr>
<td>EGM08 to 360, RTM 30’</td>
<td>-.128</td>
<td>0.030</td>
</tr>
<tr>
<td>EGM08 to 2190, RTM 5’</td>
<td>-.114</td>
<td>0.036</td>
</tr>
<tr>
<td>EGM08 to 2190, RTM 30’</td>
<td>-.138</td>
<td>0.029</td>
</tr>
</tbody>
</table>

Fig. 6: Left: Differences between GPS-leveling and geoid in the central region in cm, with the geoid model (EGM08-2190, colour range 46 to 53 m) in background. Right: Distribution of gravity points (crosses) and GPS-leveling points (circles) in central region; colours show DEM heights (colour scale 0 to 1800 m)

To judge the errors of the GPS-leveling comparison in details, Figure 6 shows the geoid outlier values in cm, as well as the underlying gravity data coverage in the central area. This coverage is relatively good, although parts of the area have
gaps of 5 km extent or more, a possible source of some of the discrepancy between geoid and GPS.

To further check for systematic errors, Fig. 7 shows the correlation of the differences with height; the lack of such correlation confirms a high-quality geoid solution and been obtained, and has been compared to high-quality GPS-levelling data. Overall, however, the agreement between GPS-leveling and geoid is probably as good as one could expect, given the earlier quoted 2-3 cm rms accuracy of the GPS-leveling. The experiment thus confirms that a very precise geoid computation and validation can be done in the Auvergne area by GRAVSOFT.

![Graph showing distribution of outliers for the gravimetric geoid](image)

**Fig. 7:** Distribution of outliers for the gravimetric geoid (using EGM08 to degree 2190) as a function of height

5. Conclusions

The computations in the Auvergne test area confirms that it is possible to carry out a geoid determination fitting at the 3 cm r.m.s. level to high-quality GPS-levelling, and that the residuals are uncorrelated with height. In reality it is not really possible to estimate the actual accuracy of the geoid, since the comparison error is close to the estimated errors of the GPS-leveling.

It is also shown that the EGM08 in itself provides an excellent fit to the geoid of the Auvergne region, illustrating that when the underlying data are good, EGM08 also provides excellent results. Combining EGM08 with the RTM method, it is also seen that some theoretical errors prevent the fully consistent use of residual topography, when implemented in space domain as prism computations. It should therefore be avoided to use too high-resolution reference elevation surfaces in RTM with GRAVSOFT.

For a comparison of the results of this paper with other results in the region, as given in Barzaghi and Sanso (2009), the 29 mm r.m.s. fit of the best results compare very well with the results of other methods; the 6 other groups reporting r.m.s.
comparing results in the range of 29 to 65 mm, with a median result of 35 mm r.m.s. Some possible theoretical improvements to the GRAVSOFT implementation, e.g. using improved higher-order atmospheric corrections, series expansions of the harmonic correction, and a rigorous use of $T_{zz}$-gradients in Molodensky harmonic continuation, could make results better. However, the ultimate 1 cm geoid in the mountains can probably not be attained until more gravity data is available, and the terrain reduction is done using improved density estimates.

References


