Triaxial coordinate systems and their geometrical interpretation

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Abstract: Coordinate systems on a triaxial ellipsoid are summarized and presented, together with their geometrical interpretation. Firstly, the geodetic coordinate system is a generalization of the geodetic system on a biaxial ellipsoid. Then, the ellipsoidal coordinate system is a triply orthogonal system, related to the theory of potential. Finally, the geometric coordinate system, which is based on the previous systems, is interpreted. The three coordinate systems on a biaxial ellipsoid are obtained as degenerate cases. In addition, a method to compute the three curvilinear coordinates of each system is described.

1. Introduction

It is generally accepted in the geodetic community that a triaxial ellipsoid could better approximate the geoid than the biaxial ellipsoid. Furthermore, several non-spherical celestial bodies such as planets, natural satellites, asteroids and comets are modeled by a triaxial ellipsoid. Also, present day accuracy requirements and modern computational capabilities encourage the study of the triaxial ellipsoid as a geometrical and a physical model in geodesy and related sciences.

From a geometrical viewpoint, Shebl and Farag (2007) and Panou (2013) presented methods for the solution of the geodesic problem on a triaxial ellipsoid. Also, Klein (2012) gave a solution to the problem of the intersection of an ellipsoid and a plane. Furthermore, various map projections have been developed on a triaxial ellipsoid (Weightman 1961, Snyder 1985, Grafarend and Krumm 2006, Fleis et al. 2013, Nyrtsov 2014). The geodetic (planetographic) coordinate system on a triaxial ellipsoid has been presented by Grafarend and Krumm (2006) and recently by Feltens (2009), Ligas (2012a, b) and Bektaş (2014). On the other hand, the ellipsoidal coordinate system which is used in the classical textbooks of potential theory (Hobson 1931, Kellogg 1953, MacMillan 1958, Pick et al. 1973, Dassios 2012) and in works of Balmino (1994), Miloh (1990), Garmier and Barriot (2001), Hu (2012) and Lowes and Winch (2012) has disadvantages. To overcome these problems, Tabanov (1999) introduced new ellipsoidal coordinates and subsequently Panou (2014a, b) gave their geometrical interpretation. However, Caputo (1967) has developed the geometric coordinate system. The present paper reviews the triaxial coordinate systems, including their geometrical interpretation.

From a physical viewpoint, research on the theory of ellipsoidal figures of equilibrium began with Newton and has continued to the present day. The monograph by Chandrasekhar (1969) summarizes much of this research as of 1968, while some of the developments of the next thirty years are included in the work of Lebovitz (1998).

2. Geodetic coordinates (h, φ, λ)

In order to introduce a triaxial coordinate system, we consider a triaxial ellipsoid which, in Cartesian coordinates (x, y, z) is described by

$$S_0 : \frac{x^2}{a_x^2} + \frac{y^2}{a_y^2} + \frac{z^2}{b^2} = 1 , \tag{1}$$

where $b < a_y < a_x$ are its three semiaxes (a_x = major equatorial semiaxis, a_y = minor equatorial semiaxis, b = polar semiaxis). This ellipsoid has three principal ellipses, mutually perpendicular. For each ellipse, we can calculate the first eccentricity

$$e_x = \sqrt{a_x^2 - b^2} \Big/ a_x , \quad e_y = \sqrt{a_y^2 - b^2} \Big/ a_y , \quad e_e = \sqrt{a_x^2 - a_y^2} \Big/ a_x .$$

In a Cartesian coordinate system, a point P outside or inside of a triaxial ellipsoid has the coordinates (x, y, z). Similarly to a biaxial ellipsoid, the geodetic height h is the distance along the surface normal from the ellipsoid S_0 to the point P (see Figure 1). The geodetic latitude φ and geodetic longitude λ are related to the ellipsoidal normal unit vector $\mathbf{n} = (\cos\varphi \cos\lambda , \cos\varphi \sin\lambda , \sin\varphi)$. Introducing the radius of curvature N of the prime vertical section in the components of this vector, the Cartesian coordinates of point Q on the triaxial ellipsoid are expressed:

$$(x, y, z) = (N \cos\varphi \cos\lambda , \; N\left(1 - e_e^2\right)\cos\varphi \sin\lambda , \; N\left(1 - e_x^2\right)\sin\varphi). \tag{2}$$

Substituting these expressions into (1), we obtain

$$N = a_x \Big/ \sqrt{1 - e_x^2 \sin^2 \varphi - e_e^2 \cos^2 \varphi \sin^2\lambda} . \tag{3}$$

Also, inverting (2), we have

$$\varphi = \tan^{-1}\left[\frac{z}{\sqrt{\left(1 - e_x^2\right)^2 x^2 + \left(1 - e_y^2\right)^2 y^2}} \right], \tag{4}$$

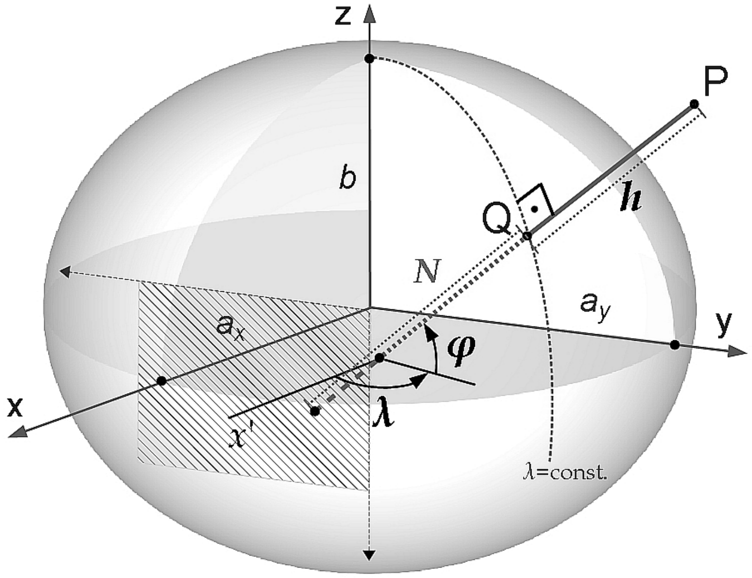

Fig. 1. *Geodetic coordinates*

$$\lambda = \tan^{-1}\left[\frac{\left(1-e_y^2\right)}{\left(1-e_x^2\right)}\frac{y}{x}\right]. \tag{5}$$

On the other hand, the ellipsoidal normal vector can be calculated by applying the gradient operator in (1), i.e. $\boldsymbol{\eta} = 2(x/a_x^2 ,\ y/a_y^2 ,\ z/b^2)$. Using the first eccentricities, we obtain three equivalent expressions for the ellipsoidal normal vector:

$$\boldsymbol{\eta}_1 = (\eta_{1x}, \eta_{1y}, \eta_{1z}) = (x\left(1-e_x^2\right),\ y\left(1-e_y^2\right), z), \tag{6}$$

$$\boldsymbol{\eta}_2 = (\eta_{2x}, \eta_{2y}, \eta_{2z}) = (x\left(1-e_e^2\right), y,\ z/\left(1-e_y^2\right)), \tag{7}$$

$$\boldsymbol{\eta}_3 = (\eta_{3x}, \eta_{3y}, \eta_{3z}) = (x,\ y/\left(1-e_e^2\right),\ z/\left(1-e_x^2\right)). \tag{8}$$

As pointed out in Feltens (2009), ellipsoidal normal vectors $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2$, $\boldsymbol{\eta}_3$ connect point Q on the ellipsoid with the $z = 0$, $y = 0$, $x = 0$ planes, respectively. Also, vector $\boldsymbol{\eta}_3$ is the longest one, with magnitude equal to the radius of curvature of the prime vertical section, i.e. $\left|\boldsymbol{\eta}_3\right| = N$.

From (4) and (6), the geodetic latitude φ is interpreted as the angle between the ellipsoidal normal vector and the projection of this vector onto the $z = 0$ plane (Figure 1), i.e. $\varphi = \tan^{-1}\left(\eta_{1z}\big/\sqrt{\eta_{1x}^2 + \eta_{1y}^2}\right)$. Similarly, from (5) and (6), the geodetic longitude λ is interpreted as the angle, measured in the $z = 0$ plane, between a line parallel to the x-axis and the projection of the ellipsoidal normal vector onto the $z = 0$ plane (Figure 1), i.e. $\lambda = \tan^{-1}\left(\eta_{1y}/\eta_{1x}\right)$.

Introducing the geodetic height h into (2), the geodetic coordinates (h, φ, λ) are related to the corresponding Cartesian coordinates (x, y, z) by

$$x = \left(N + h\right)\cos\varphi \cos\lambda\,, \tag{9}$$

$$y = \left[N\left(1 - e_e^2\right) + h\right]\cos\varphi \sin\lambda\,, \tag{10}$$

$$z = \left[N\left(1 - e_x^2\right) + h\right] \sin\varphi\,, \tag{11}$$

where $h \geq 0$, $-\pi/2 \leq \varphi \leq +\pi/2$, $-\pi < \lambda \leq +\pi$. Also, when $a_x = a_y \equiv a$, i.e. $e_x = e_y \equiv e$ and $e_e = 0$, (9)-(11) reduce to the well-known geodetic system on a biaxial ellipsoid (e.g. Fotiou 2007).

3. Ellipsoidal coordinates (u, β, ω)

A family of confocal quadrics (second degree surfaces) to the ellipsoid S_0 is given as

$$S_s:\ \frac{x^2}{a_x^2 + s} + \frac{y^2}{a_y^2 + s} + \frac{z^2}{b^2 + s} = 1\,, \tag{12}$$

where s is a real number called the parameter of the family. In Dassios (2012) is proved that, for every point (x, y, z) in space with $xyz \neq 0$ (this excludes the Cartesian planes $x = 0$, $y = 0$ and $z = 0$) Eq. (12), which is a cubic equation in s, has three unequal real roots s_1, s_2, s_3 such that

$$-a_x^2 < s_3 < -a_y^2 < s_2 < -b^2 < s_1 < +\infty\,. \tag{13}$$

Thus, through each point (x, y, z) in space with $xyz \neq 0$ passes exactly one triaxial ellipsoid (s_1 = const.), one hyperboloid of one sheet (s_2 = const.) and one hyperboloid of two sheets (s_3 = const.). These variables (s_1, s_2, s_3) are known as ellipsoidal coordinates and have dimensions of length squared. Also, the ellipsoidal coordinate system (s_1, s_2, s_3) is a triply orthogonal system and the principal sections (see below) of the coordinate surfaces share three pairs of foci: ($\pm\ E_x$, 0, 0), ($\pm\ E_e$, 0, 0), (0, $\pm\ E_y$, 0), where $E_x = \sqrt{a_x^2 - b^2}$, $E_y = \sqrt{a_y^2 - b^2}$ and

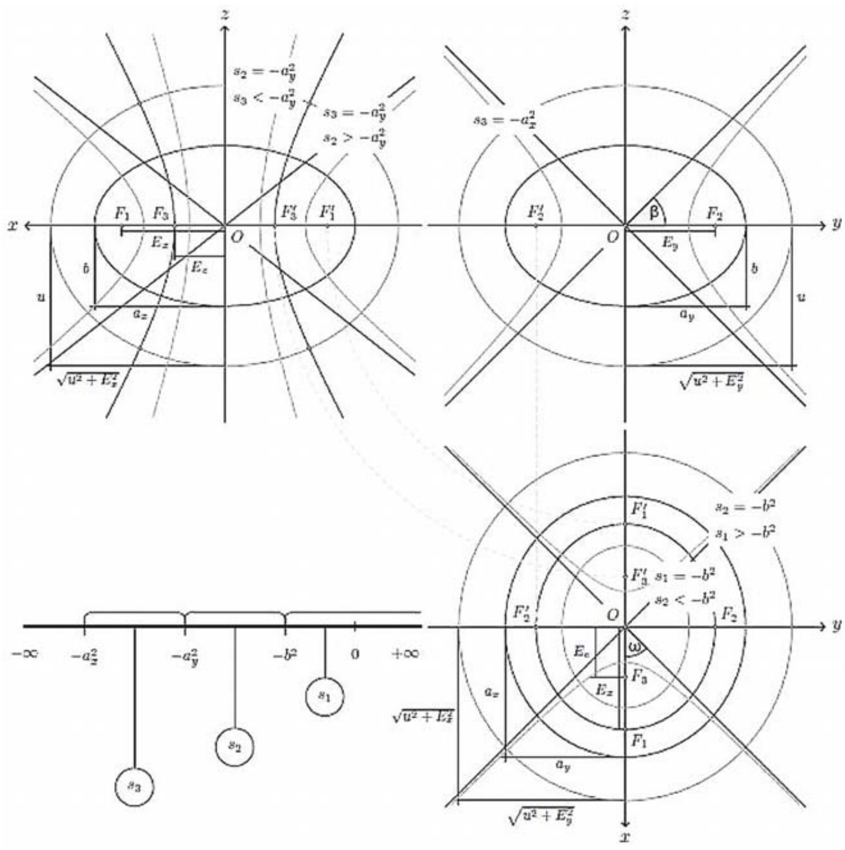

Fig. 2. *Ellipsoidal coordinates and Cartesian planes y = 0 (top left), x = 0 (top right) and z = 0 (bottom right). Illustration of Eq. (13) is also shown (bottom left)*

$E_e = \sqrt{a_x^2 - a_y^2}$ are the linear eccentricities (see Figure 2).

Figure 2 displays the Cartesian planes $x = 0$, $y = 0$ and $z = 0$. These planes intersect any one of the confocal quadrics either in an ellipse or in a hyperbola, which are called principal ellipses and principal hyperbolas of the corresponding quadric. From (12), a family of confocal principal hyperbolas is obtained

$$\frac{y^2}{a_y^2 + s_2} + \frac{z^2}{b^2 + s_2} = 1, \quad x = 0, \tag{14}$$

with foci at $(0, \pm E_y, 0)$. The linear equation

$$z = \pm y \frac{\sqrt{-b^2 - s_2}}{\sqrt{a_y^2 + s_2}}, \tag{15}$$

represents the two asymptotes of the family of hyperbolas. Also from (12), another family of confocal principal hyperbolas is obtained

$$\frac{x^2}{a_x^2 + s_3} + \frac{y^2}{a_y^2 + s_3} = 1, \quad z = 0, \tag{16}$$

with foci at ($\pm\ E_e$, 0, 0). The linear equation

$$y = \pm x \frac{\sqrt{-a_y^2 - s_3}}{\sqrt{a_x^2 + s_3}}, \tag{17}$$

represents the two asymptotes of the family of hyperbolas. Note that the confocal hyperboloids of two sheets do not intersect the plane $x = 0$. Finally, when the ellipsoidal coordinates (s_1, s_2, s_3) reach their limiting values, we get degenerate quadrics corresponding to parts of the planes $x = 0$, $y = 0$ and $z = 0$ (see Figure 2).

According to (12) there are, in general, eight points ($\pm x$, $\pm y$, $\pm z$) in space, symmetrically located in octants, corresponding to the same (s_1, s_2, s_3). Thus, in order to have a one-to-one correspondence between ellipsoidal and Cartesian coordinates, Panou (2014a, b) introduced ellipsoidal coordinates (u, β, ω) by the relations

$$s_1 = u^2 - b^2, \tag{18}$$

$$s_2 = -a_y^2 \sin^2 \beta - b^2 \cos^2 \beta, \tag{19}$$

$$s_3 = -a_x^2 \sin^2 \omega - a_y^2 \cos^2 \omega. \tag{20}$$

We pass through P a triaxial ellipsoid whose centre is the origin O, its polar axis coincides with the z-axis, its major equatorial axis coincides with the x-axis, its minor equatorial axis coincides with the y-axis and two linear eccentricities have the constant values E_x and E_y as above. The coordinate u is the polar semiaxis of this ellipsoid, β is the ellipsoidal latitude and ω is the ellipsoidal longitude. Substituting (19) into (15), we obtain

$$\beta = \tan^{-1}\left(\frac{z}{y}\right), \tag{21}$$

which leads to the interpretation that the ellipsoidal latitude β characterizes the inclination of the asymptotes of the family of confocal principal hyperbolas (14) on the plane $x = 0$. Similarly, substituting (20) into (17), we obtain

$$\omega = \tan^{-1}\left(\frac{y}{x}\right), \tag{22}$$

thus the ellipsoidal longitude ω characterizes the inclination of the asymptotes of the family of confocal principal hyperbolas (16) on the plane $z = 0$ (Figure 2).

Substituting (18)-(20) into (12), we derive the equations introduced by Tabanov (1999) and presented also by Dassios (2012)

$$x = \sqrt{u^2 + E_x^2}\left(\cos^2\beta + \frac{E_e^2}{E_x^2}\sin^2\beta\right)^{1/2}\cos\omega\,, \tag{23}$$

$$y = \sqrt{u^2 + E_y^2}\cos\beta\sin\omega\,, \tag{24}$$

$$z = u\sin\beta\left(1 - \frac{E_e^2}{E_x^2}\cos^2\omega\right)^{1/2}, \tag{25}$$

where $u \geq 0$, $-\pi/2 \leq \beta \leq +\pi/2$, $-\pi < \omega \leq +\pi$. Also, when $a_x = a_y \equiv a$, i.e. $E_x = E_y \equiv E$ and $E_e = 0$, (23)-(25) reduce to the well-known oblate spheroidal system (e.g. Heiskanen and Moritz 1967).

4. Geometric coordinates (τ, θ, ε)

Formulas relating geometric (τ, θ, ε) and Cartesian (x, y, z) coordinates are introduced by Caputo (1967):

$$x = \frac{\tau^2 + E_x^2}{d}\cos\theta\cos\varepsilon\,, \tag{26}$$

$$y = \frac{\tau^2 + E_y^2}{d}\cos\theta\sin\varepsilon\,, \tag{27}$$

$$z = \frac{\tau^2}{d}\sin\theta\,, \tag{28}$$

where $\tau \geq 0$, $-\pi/2 \leq \theta \leq +\pi/2$, $-\pi < \varepsilon \leq +\pi$ and

$$d = \sqrt{a_x^2\cos^2\theta\cos^2\varepsilon + a_y^2\cos^2\theta\sin^2\varepsilon + b^2\sin^2\theta + \tau^2 - b^2}\,. \tag{29}$$

Also, when $a_x = a_y \equiv a$, i.e. $E_x = E_y \equiv E$ and $E_e = 0$, (26)-(29) reduce to the geometric system on a biaxial ellipsoid.

In order to give the geometrical interpretation, we pass through P an ellipsoid

$$S_{\tau^2-b^2}: \frac{x^2}{\tau^2 + E_x^2} + \frac{y^2}{\tau^2 + E_y^2} + \frac{z^2}{\tau^2} = 1\,. \tag{30}$$

Substituting (26)-(28) into (30), we can easily derive (29). Thus, τ is the polar semiaxis of the ellipsoid $S_{\tau^2-b^2}$. The geometric latitude θ is the angle between the normal to the ellipsoid $S_{\tau^2-b^2}$ containing the point P and the projection of this normal onto the $z = 0$ plane. The geometric longitude ε is the angle between a line parallel to the x-axis and the projection of the normal to the ellipsoid $S_{\tau^2-b^2}$ containing the point P onto the $z = 0$ plane (Figure 3).

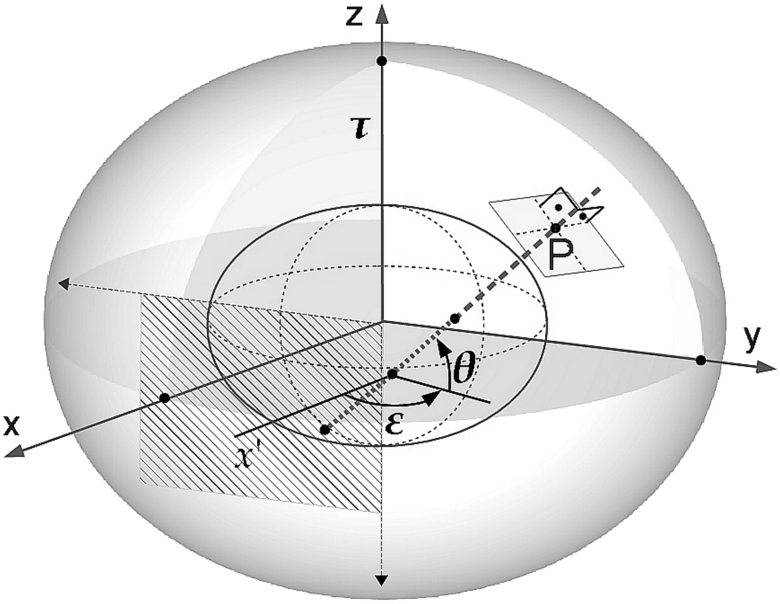

Fig. 3. *Geometric coordinates*

This interpretation follows easily, considering the normal $\boldsymbol{\sigma}$ to the ellipsoid $S_{\tau^2-b^2}$

$$\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z) = 2(x/\left(\tau^2 + E_x^2\right),\ y/\left(\tau^2 + E_y^2\right),\ z/\tau^2), \tag{31}$$

and (26)-(28), i.e.

$$\theta = \tan^{-1}\left(\frac{\sigma_z}{\sqrt{\sigma_x^2 + \sigma_y^2}}\right) = \tan^{-1}\left[\frac{\left(\tau^2 + E_x^2\right)\left(\tau^2 + E_y^2\right)z}{\tau^2\sqrt{\left(\tau^2 + E_y^2\right)^2 x^2 + \left(\tau^2 + E_x^2\right)^2 y^2}}\right], \tag{32}$$

$$\varepsilon = \tan^{-1}\left(\frac{\sigma_y}{\sigma_x}\right) = \tan^{-1}\left[\frac{\left(\tau^2 + E_x^2\right)}{\left(\tau^2 + E_y^2\right)}\frac{y}{x}\right]. \tag{33}$$

5. Concluding remarks

In this work, three triaxial coordinate systems have been reviewed, along with their geometrical interpretation. Feltens (2009), Ligas (2012a, b) and Bektaş (2014) have presented a numerical method to compute the projection Q of a point P along the normal to a triaxial ellipsoid. Then, we can compute the geodetic height h by the Euclidean distance and the geodetic latitude φ and longitude λ by applying (4) and (5), respectively. On the other hand, substituting the known Cartesian coordinates (x, y, z) of point P in (12), we obtain a cubic equation in s, from which we can evaluate the three real roots s_1, s_2 and s_3. Subsequently, we use (18)-(20) to compute the ellipsoidal coordinates (u, β, ω). Finally, substituting the known Cartesian coordinates (x, y, z) in (32) and (33), we compute the geometric latitude θ and geometric longitude ε, respectively. The coordinate τ is computed using (12) and the parameter s_1.

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